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# Stability of finite-dimensional nonlinear elastic systems with unilateral contact and friction

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**Abstract**

This paper addresses some questions in the general area of the instability of finite-dimensional elastic systems in unilateral frictional contact with rigid obstacles. We study the occurrence of dynamic solutions in the neighborhood of a given equilibrium state which might tend to diverge from that state. Some of the results obtained by Martins et al. (1998) are generalized here to encompass the effects of the system *nonlinear elastic behavior* and of the *obstacle curvature*. © 2000 Elsevier Science Ltd. All rights reserved.

## 1. Introduction

As discussed in Martins et al. (1998), no dynamic solution may be initiated at an equilibrium configuration of a finite-dimensional scleronomic frictional contact system with a velocity discontinuity and an impulsive reaction. This is so because the kinetic energy of the system is a positive definite quadratic form on the system generalized velocities that already has its minimum value (zero) at the equilibrium configuration, and all physically admissible velocity discontinuities must correspond to a non-positive jump in kinetic energy. The questions addressed in this paper are thus the following:

(i) the occurrence of dynamic solutions initiating at an equilibrium position with no initial perturbation but with an initial acceleration and reaction discontinuity; this is a mass and friction induced phenomenon of non-uniqueness of dynamic solutions;

(ii) the occurrence of divergence instabilities of equilibrium states, i.e. the existence of smooth non-oscillatory growing dynamic solutions with perturbed initial conditions arbitrarily close to an equilibrium configuration.

In what concerns the first topic, we observe that the related problem of computing the accelerations of a multi-degree-of-freedom system with frictional unilateral contacts has been addressed by Lötstedt

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(1981) and by Glocker and Pfeiffer (1992, 1993) in the plane case, and by Pang and Trinkle (1996) and Trinkle et al. (1997) in the three dimensional case. The theory of linear complementarity or the theory of quasivariational inequalities were used by those authors. Conditions for existence and uniqueness of solution were proved by Lötstedt (1981), Pang and Trinkle (1996) and Trinkle et al. (1997).

In the present paper we concentrate on what can happen at an equilibrium configuration of the system (its initial velocity is null): *necessary* and *sufficient* conditions are established in Section 3 for the occurrence of dynamic solutions initiating at one such equilibrium configuration with an initial acceleration and reaction discontinuity. These conditions can be expressed in various alternative forms, which result from different formulations for the problem of determining, at the equilibrium configuration, the admissible (right) accelerations and reactions of the system: inclusion and variational statements are discussed in the present paper. An example problem from the literature, where mathematical difficulties are known to arise due to combined effects of inertia and friction, is discussed here having in mind these initial acceleration and reaction discontinuities.

In what concerns the second topic, we observe that studies on bifurcations and (divergence) instabilities in continuum elastic systems with unilateral contacts were performed by Chateau and Nguyen (1989), in the frictionless case, and by Nguyen (1990) and Chateau and Nguyen (1991), in the frictional case; a necessary condition for the occurrence of divergence instabilities was proved in the latter work. Klarbring (1988) analysed the stability of discrete nonlinear elastic systems with frictionless contacts; Björkman (1992) studied the occurrence of bifurcation, limit and end points in quasistatic trajectories involving also discrete nonlinear elastic systems with frictionless contacts. Mróz and Plaut (1992) studied the stability of finite-dimensional systems with friction, but with constant normal reactions, which makes the friction forces derivable from a potential and eliminates the non-associative character of the most general frictional contact problems.

In Section 4 of the present paper, *sufficient* conditions are established for the occurrence of divergence instabilities of equilibrium states, in finite-dimensional nonlinear elastic systems with unilateral contacts where the non-associative friction law of Coulomb holds. Such conditions involve the current mass and stiffness properties of the system, as well as the current normal and tangential state of the contact candidate particles. The analysis leads to the study of dynamic stability *eigenproblems* for which inclusion or variational inequality statements are given. The construction of such problems is discussed and illustrated in two examples of small dimension where the instabilizing effects of Coulomb's friction are combined with those of geometric nonlinearity or of obstacle curvature. In particular, an example problem is discussed which has the characteristic behavior of what might be called a *non-associative Shanley column*.

We start by introducing in Section 2 the notation and some preliminary results needed for the development of the above topics.

## 2. Formulation and preliminary results

### 2.1. Dynamic and static contact problems with friction

We consider a plane holonomic and scleronomic finite-dimensional mechanical system whose configuration at each time $t \geqslant 0$ is described by the values $X_i(t)$, $1 \leqslant i \leqslant N$, of the independent generalized coordinates; the corresponding column vector of the values at time $t$ of those generalized coordinates is denoted by $\mathbf{X}(t) \in \mathbb{R}^N$. A finite number of particles of that mechanical system is subjected to unilateral contact constraints with fixed curved obstacles. The set $P_C \subset \mathbb{N}$ groups the labels of the particles (p) of those Contact candidate particles.

Each point in the plane of the system is identified by the column vector $\mathbf{x}$ of the components $x_\alpha$, $\alpha = 1, 2$, of its position vector in some fixed orthonormal reference frame $(O, \mathbf{e}_1, \mathbf{e}_2)$. In this paper, Greek subscripts ($\alpha, \beta, \ldots = 1, 2$) will be used always to denote the Cartesian components of vectors of the plane of the system in the same reference frame. For each contact candidate particle p, the corresponding obstacle is identified by the set of vectors $\mathbf{x} \in \mathbb{R}^2$ such that

$$\varphi^{\mathrm{p}}(\mathbf{x}) = 0, \tag{2.1}$$

where the function $\varphi^{\mathrm{p}}: \mathbb{R}^2 \rightarrow \mathbb{R}$ is at least twice continuously differentiable and

$$\left| \frac{\partial \varphi^{\mathrm{p}}}{\partial \mathbf{x}} \right| \neq 0 \tag{2.2}$$

holds at the points on or sufficiently close to the obstacle. On each point of these obstacles, the unit normal and tangent vectors are defined, respectively, by

$$\mathbf{n}^{\mathrm{p}}(\mathbf{x}) \stackrel{\mathrm{def}}{=} \frac{\partial \varphi^{\mathrm{p}}/\partial \mathbf{x}}{|\partial \varphi^{\mathrm{p}}/\partial \mathbf{x}|}(\mathbf{x}), \quad \mathbf{t}^{\mathrm{p}}(\mathbf{x}) \stackrel{\mathrm{def}}{=} (\mathbf{e}_1 \times \mathbf{e}_2) \times \mathbf{n}^{\mathrm{p}}(\mathbf{x}), \tag{2.3}$$

and the obstacle curvature is given by

$$\chi^{\mathrm{p}}(\mathbf{x}) = -\frac{1}{\left| \frac{\partial \varphi^{\mathrm{p}}}{\partial \mathbf{x}}(\mathbf{x}) \right|} \frac{\partial^2 \varphi^{\mathrm{p}}}{\partial x_\alpha \, \partial x_\beta}(\mathbf{x}) t_\alpha^{\mathrm{p}}(\mathbf{x}) t_\beta^{\mathrm{p}}(\mathbf{x}). \tag{2.4}$$

In this formula the summation convention applies to Greek subscripts and throughout this paper it also applies to the Latin indices $i, j, \ldots = 1, \ldots, N$. Note that the summation convention does not apply to the superscript p that denotes the contact candidate particles. In view of the assumptions above, the definition (2.3) of the orthonormal basis $(\mathbf{n}^{\mathrm{p}}(\mathbf{x}), \mathbf{t}^{\mathrm{p}}(\mathbf{x}))$ may be extended to all points of the plane that are sufficiently close to the obstacle p.

The position of each particle $\mathrm{p} \in P_{\mathrm{C}}$ at each time $t \geqslant 0$ is identified by the column vector $\mathbf{x}^{\mathrm{p}}(t) = \mathbf{x}^{\mathrm{p}}(\mathbf{X}(t)) \in \mathbb{R}^2$, and the column vector of the normal and tangential components of the particle velocity is given by

$$\mathbf{v}^{\mathrm{p}}(t) = \begin{bmatrix} v_{\mathrm{n}}^{\mathrm{p}}(t) \\ v_{\mathrm{t}}^{\mathrm{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{\mathrm{n}}^{\mathrm{p}}(\mathbf{X}(t)) \\ \mathbf{G}_{\mathrm{t}}^{\mathrm{p}}(\mathbf{X}(t)) \end{bmatrix} \dot{\mathbf{X}}(t) = \mathbf{G}^{\mathrm{p}}(\mathbf{X}(t)) \dot{\mathbf{X}}(t), \tag{2.5}$$

where the $(1 \times N)$ row matrices $\mathbf{G}_{\mathrm{n}}^{\mathrm{p}}(\mathbf{X})$ and $\mathbf{G}_{\mathrm{t}}^{\mathrm{p}}(\mathbf{X})$ have the components

$$G_{\mathrm{n}i}^{\mathrm{p}}(\mathbf{X}) = \mathbf{n}^{\mathrm{p}}(\mathbf{x}^{\mathrm{p}}(\mathbf{X})) \cdot \frac{\partial \mathbf{x}^{\mathrm{p}}}{\partial \mathbf{X}_i}(\mathbf{X}), \quad G_{\mathrm{t}i}^{\mathrm{p}}(\mathbf{X}) = \mathbf{t}^{\mathrm{p}}(\mathbf{x}^{\mathrm{p}}(\mathbf{X})) \cdot \frac{\partial \mathbf{x}^{\mathrm{p}}}{\partial \mathbf{X}_i}(\mathbf{X}), \quad i = 1, \ldots, N. \tag{2.6}$$

As usual, the notation $(\dot{\ })$ denotes the time derivative $\mathrm{d}(\ )/\mathrm{d}t$. The velocities (2.5) of the contact candidate particles are grouped in a single column vector $\mathbf{v}(t)$ of dimension $2n_{\mathrm{C}} \times 1 (n_{\mathrm{C}} = \# P_{\mathrm{C}})$ and, accordingly, the $2n_{\mathrm{C}} \times N$ matrix $\mathbf{G}(\mathbf{X})$ is constructed such that

$$\mathbf{v}(t) = \mathbf{G}(\mathbf{X}(t)) \dot{\mathbf{X}}(t). \tag{2.7}$$

We denote by $\mathbf{r}^{\mathrm{p}}(t) = [r_{\mathrm{n}}^{\mathrm{p}}(t) \quad r_{\mathrm{t}}^{p}(t)]^{\mathrm{T}}$ the column vector of the normal and tangential components of the reaction force that acts at some time $t \geqslant 0$ on the contact candidate particle $p \in P_{\mathrm{C}}$. The column vector (of dimension $2n_{\mathrm{C}} \times 1$) that groups all the reaction vectors $\mathbf{r}^{\mathrm{p}}(t)$ is denoted by $\mathbf{r}(t)$. For some contact reactions $\mathbf{r}(t) \in \mathbb{R}^{2n_{\mathrm{C}}}$ at some configuration $\mathbf{X}(t)$ of the system, the vector of generalized reactions $\mathbf{R}(t) \in \mathbb{R}^N$ is given by

$$\mathbf{R}(t) = \mathbf{G}^{\mathrm{T}}(\mathbf{X}(t))\mathbf{r}(t). \tag{2.8}$$

**Remark 2.1.** Whenever confusion is not likely to arise, the functional dependence of some quantities on the configuration **X** will be omitted. In particular, the matrix **G(X)** will be frequently denoted simply by **G**, and notations of the type in (2.5), $\mathbf{G}_{\mathrm{n}}^{\mathrm{p}}(\mathbf{X})\dot{\mathbf{X}}$ and $\mathbf{G}_{\mathrm{t}}^{\mathrm{p}}(\mathbf{X})\dot{\mathbf{X}}$, will be abbreviated to $\gamma_{\mathrm{n}}^{\mathrm{p}}(\dot{\mathbf{X}})$ and $\gamma_{\mathrm{t}}^{\mathrm{p}}(\dot{\mathbf{X}})$, respectively. In some circumstances, the dependency on the contact particle p will also be omitted, and $r_{\mathrm{n}}^{\mathrm{p}}$, $r_{\mathrm{t}}^{\mathrm{p}}$, $\gamma_{\mathrm{n}}^{\mathrm{p}}(\dot{\mathbf{X}})$ and $\gamma_{\mathrm{t}}^{\mathrm{p}}(\dot{\mathbf{X}})$ will be denoted simply by $r_{\mathrm{n}}$, $r_{\mathrm{t}}$, $\gamma_{\mathrm{n}}(\dot{\mathbf{X}})$ and $\gamma_{\mathrm{t}}(\dot{\mathbf{X}})$, respectively. □

**Remark 2.2.** In the following it will be frequently assumed that:

*the lines of the* $2n_{\mathrm{C}} \times N$ *matrix* **G** *are linearly independent.* (2.9)

In these circumstances the linear map $\mathbf{G}^{\mathrm{T}}: \mathbb{R}^{2n_{\mathrm{C}}} \longrightarrow \mathbb{R}^{N}$ has a left inverse, which can be represented by the $2n_{\mathrm{C}} \times N$ matrix:

$$\mathbf{G}^{\dagger} = (\mathbf{G}\mathbf{G}^{\mathrm{T}})^{-1}\mathbf{G}, \tag{2.10}$$

which means that $\mathbf{G}^{\dagger}\mathbf{G}^{\mathrm{T}}\mathbf{r} = \mathbf{r}$ for all $\mathbf{r} \in \mathbb{R}^{2n_{\mathrm{C}}}$, and that $\mathbf{G}^{\mathrm{T}}\mathbf{G}^{\dagger}\mathbf{R} = \mathbf{R}$ for all $\mathbf{R} \in Rg(\mathbf{G}^{\mathrm{T}}) \subset \mathbb{R}^{N}$ (Lancaster and Tismenetsky, 1985).

For each particle $\mathrm{p} \in P_{\mathrm{C}}$, the normal and tangential reactions corresponding to the generalized reaction $\mathbf{R} \in Rg(\mathbf{G}^{\mathrm{T}})$ are then obtained by $r_{\mathrm{n}}^{\mathrm{p}} = \mathbf{G}_{\mathrm{n}}^{\dagger\mathrm{p}}\mathbf{R}$ and $r_{\mathrm{t}}^{\mathrm{p}} = \mathbf{G}_{\mathrm{t}}^{\dagger\mathrm{p}}\mathbf{R}$, where $\mathbf{G}_{\mathrm{n}}^{\dagger\mathrm{p}}$ and $\mathbf{G}_{\mathrm{t}}^{\dagger\mathrm{p}}$ are the appropriate $(1 \times N)$ row submatrices of $\boldsymbol{G}^{\dagger}$. Accordingly, the column vector **r** is computed by

$$\mathbf{r} = \mathbf{G}^{\dagger}\mathbf{R}. \tag{2.11}$$

Similarly to Remark 2.1 and as done above, the **X** dependency will be frequently omitted from the notation and the p dependency will also be omitted in some circumstances, so that $\mathbf{G}_{\mathrm{n}}^{\dagger\mathrm{p}}(\mathbf{X})\mathbf{R}$ and $\mathbf{G}_{\mathrm{t}}^{\dagger\mathrm{p}}(\mathbf{X})\mathbf{R}$ will be frequently denoted by $\gamma_{\mathrm{n}}^{\dagger\mathrm{p}}(\mathbf{R})$ and $\gamma_{\mathrm{t}}^{\dagger\mathrm{p}}(\mathbf{R})$, or simply by $\gamma_{\mathrm{n}}^{\dagger}(\mathbf{R})$ and $\gamma_{\mathrm{t}}^{\dagger}(\mathbf{R})$, respectively. □

The mechanical system is assumed to be nonlinear elastic with a strain energy $\mathrm{U} = \mathrm{U}(\mathbf{X})$, and is acted upon by constant external applied forces such that $\Omega = \Omega(\mathbf{X})$ is the corresponding potential energy. $\mathbf{F}^{\mathrm{U}}(\mathbf{X})$ is the vector of the generalized elastic forces and $\mathbf{F}^{\Omega}(\mathbf{X})$ is the vector of the generalized external forces:

$$F_i^{\mathrm{U}}(\mathbf{X}) = -\frac{\partial \mathrm{U}}{\partial X_i}(\mathbf{X}), \quad F_i^{\Omega}(\mathbf{X}) = -\frac{\partial \Omega}{\partial X_i}(\mathbf{X}), \quad i = 1, \ldots, N. \tag{2.12}$$

We denote by $\mathrm{T}(\mathbf{X}, \dot{\mathbf{X}})$ the kinetic energy of the system:

$$\mathrm{T}(\mathbf{X}, \dot{\mathbf{X}}) = \frac{1}{2}\mathbf{M}(\mathbf{X})\dot{\mathbf{X}} \cdot \dot{\mathbf{X}},$$

where **M(X)** is the symmetric, positive definite mass matrix.

Along portions of the system trajectory where $\mathbf{X}(t)$ is twice continuously differentiable and $\mathbf{r}(t)$ is continuous, the motion of the system is governed by the *N Lagrange equations*

$$\mathbf{M}(\mathbf{X}(t))\ddot{\mathbf{X}}(t) + \mathbf{B}(\mathbf{X}(t), \dot{\mathbf{X}}(t)) = \mathbf{G}^{\mathrm{T}}(\mathbf{X}(t))\mathbf{r}(t), \tag{2.13}$$

where

$$\mathbf{B}(\mathbf{X}, \dot{\mathbf{X}}) \overset{\mathrm{def}}{=} \mathbf{D}(\mathbf{X}, \dot{\mathbf{X}}) - \mathbf{F}^{\mathrm{U}}(\mathbf{X}) - \mathbf{F}^{\Omega}(\mathbf{X}),$$

$$D_i(\mathbf{X}, \dot{\mathbf{X}}) \stackrel{\text{def}}{=} \left[\frac{\partial M_{ij}}{\partial X_k}(\mathbf{X}) - \frac{1}{2}\frac{\partial M_{jk}}{\partial X_i}(\mathbf{X})\right]\dot{X}_j\dot{X}_k, \quad i, j, k = 1, \ldots, N. \tag{2.14}$$

Note that the vector $\mathbf{D}(\mathbf{X}, \dot{\mathbf{X}})$ groups inertia terms quadratically dependent on the generalized velocities.

In addition, the *classical unilateral contact conditions*

$$\phi^{\mathrm{p}}(\mathbf{X}(t)) \stackrel{\text{def}}{=} \varphi^{\mathrm{p}}(\mathbf{x}^{\mathrm{p}}(\mathbf{X}(t))) \leqslant 0, \quad r_{\mathrm{n}}^{\mathrm{p}}(t) \leqslant 0, \quad \phi^{\mathrm{p}}(\mathbf{X}(t))r_{\mathrm{n}}^{\mathrm{p}}(t) = 0, \tag{2.15}$$

and the *friction law of Coulomb*

$$r_{\mathrm{t}}^{\mathrm{p}}(t) \in \mu r_{\mathrm{n}}^{\mathrm{p}}(t)\sigma\left[\gamma_{\mathrm{t}}^{\mathrm{p}}(\dot{\mathbf{X}}(t))\right] \tag{2.16}$$

have to be satisfied for all contact candidate particles $p \in P_{\mathrm{C}}$; $\mu \geqslant 0$ is the coefficient of friction, $\sigma[\cdot]z$ denotes the multi-valued application such that, for each $x \in \mathbb{R}$,

$$\sigma[x] \stackrel{\text{def}}{=} \begin{cases} x/|x|, & \text{if } x \neq 0, \\ [-1,1], & \text{if } x = 0 \end{cases}. \tag{2.17}$$

The *static equilibrium* of the same mechanical system for some time independent external applied forces is a dynamic solution with null velocity and acceleration. The equilibrium is characterized by vectors of generalized coordinates and of contact reactions, $\mathbf{X}^0$ and $\mathbf{r}^0$, respectively, such that the equilibrium equations

$$\mathbf{F}^{\mathrm{U}}(\mathbf{X}^0) + \mathbf{F}^{\Omega}(\mathbf{X}^0) + \mathbf{G}^{\mathrm{T}}(\mathbf{X}^0)\mathbf{r}^0 = \mathbf{0} \tag{2.18}$$

are satisfied together with the form

$$\phi^{\mathrm{p}}(\mathbf{X}^0) \leqslant 0, \quad r_{\mathrm{n}}^{0\mathrm{p}} \leqslant 0, \quad \phi^{\mathrm{p}}(\mathbf{X}^0)r_{\mathrm{n}}^{0\mathrm{p}} = 0, \quad |r_{\mathrm{t}}^{0\mathrm{p}}| \leqslant -\mu r_{\mathrm{n}}^{0\mathrm{p}}, \tag{2.19}$$

of the unilateral frictional contact conditions at each contact candidate particle $\mathrm{p} \in P_{\mathrm{C}}$.

In what concerns the above dynamic problem, it is well known that the solutions $\mathbf{X}(t)$ are not in general twice continuously differentiable functions of time. For the reasons explained in the Introduction of this paper, we shall not discuss here the occurrence of velocity discontinuities and impulsive reactions. In order to study in Section 3 the possible occurrence of some dynamic solutions initiating at an equilibrium position with an initial acceleration and reaction discontinuity, we shall characterize in Section 2.2 the sets of admissible configurations, velocities, accelerations and reactions. In order to study in Section 4 the possible occurrence of perturbed dynamic solutions (smoothly) diverging from equilibrium, we shall characterize in Section 2.3 the sets of admissible right velocities and reaction rates.

### *2.2. Admissible configurations, velocities, accelerations and reactions*

In view of the unilateral constraints $(2.15)_1$, the *set of admissible configurations* $K_{\mathrm{X}}$ is defined by:

$$K_{\mathrm{X}} \stackrel{\text{def}}{=} \left\{\mathbf{X} \in \mathbb{R}^N : \phi^{\mathrm{p}}(\mathbf{X}) \leqslant 0, \text{ for all } \mathrm{p} \in P_{\mathrm{C}}\right\}. \tag{2.20}$$

For each $\mathbf{X} \in K_{\mathrm{X}}$ we introduce the following decomposition of the set $P_{\mathrm{C}}$ of contact candidate particles

$$P_{\rm C} = P_{\rm f}(\mathbf{X}) \cup P_{\rm c}(\mathbf{X})$$

$$P_{\rm f}(\mathbf{X}) \stackrel{\rm def}{=} \left\{ {\rm p} \in P_{\rm C}\colon \phi^{\rm p}(\mathbf{X}) < 0 \right\} \qquad \text{[particles currently not in contact (\underline{f}ree)]}$$

$$P_{\rm c}(\mathbf{X}) \stackrel{\rm def}{=} \left\{ {\rm p} \in P_{\rm C}\colon \phi^{\rm p}(\mathbf{X}) = 0 \right\} \qquad \text{[particles currently in \underline{c}ontact].}$$

Then, for each $\mathbf{X} \in K_{\rm X}$, we define the (configuration dependent) *set of admissible reaction forces*:

$$K_{\rm r}(\mathbf{X}) \stackrel{\rm def}{=} \{\mathbf{r} \in \mathbb{R}^{2n_{\rm C}}\colon \quad r_{\rm n} = r_{\rm t} = 0, \text{ in } P_{\rm f}; \quad r_{\rm n} \leqslant 0 \text{ and } |r_{\rm t}| \leqslant -\mu r_{\rm n}, \text{ in } P_{\rm c}\}. \tag{2.21}$$

Having in mind the unilateral constraints $(2.15)_1$ and observing that, for each particle currently in contact (${\rm p} \in P_{\rm c}(\mathbf{X}(t))$, $\phi^{\rm p}(\mathbf{X}(t)) = 0$), the equality

$$\frac{\rm d}{{\rm d}t}\phi^{\rm p}(\mathbf{X}(t)) = \left| \frac{\partial \varphi^{\rm p}}{\partial \mathbf{x}}(\mathbf{x}^{\rm p}(\mathbf{X}(t))) \right| v_{\rm n}^{\rm p}(t) \tag{2.22}$$

holds, we define, for each $\mathbf{X} \in K_{\rm X}$, the (configuration dependent) *set of admissible right generalized velocities*:

$$K_{\rm V}(\mathbf{X}) = \left\{ \mathbf{V} \in \mathbb{R}^N\colon \gamma_{\rm n}(\mathbf{V}) \leqslant 0, \text{ in } P_{\rm c} \right\}. \tag{2.23}$$

Having now in mind (2.15), (2.16) and (2.22), we define, for each $\mathbf{X} \in K_{\rm X}$ and each $\mathbf{V} \in K_{\rm V}(\mathbf{X})$, the (configuration and velocity dependent) *set of admissible right reaction forces*

$$\begin{aligned} K_{\rm r}(\mathbf{X}, \mathbf{V}) = \{\mathbf{r} \in \mathbb{R}^{2n_{\rm C}}\colon \quad & r_{\rm n} \leqslant 0, \ r_{\rm n}\gamma_{\rm n}(\mathbf{V}) = 0, \text{ in } P_{\rm c}; \\ & r_{\rm n} = 0, \text{ in } P_{\rm f}; \\ & r_{\rm t} \in \mu r_{\rm n}\sigma\left[\gamma_{\rm t}(\mathbf{V})\right], \text{ in } P_{\rm c}; \\ & r_{\rm t} = 0, \text{ in } P_{\rm f}\}; \\ \subset \ & K_{\rm r}(\mathbf{X}, \mathbf{0}) = K_{\rm r}(\mathbf{X}). \end{aligned} \tag{2.24}$$

For each admissible $\mathbf{X} \in K_{\rm X}$ and each $\boldsymbol{V} \in K_{\rm V}(\mathbf{X})$ we further decompose the set $P_{\rm c}(\mathbf{X})$ into

$$P_{\rm c}(\mathbf{X}) = P_{\rm cf}(\mathbf{X}, \mathbf{V}) \cup P_{\rm cc}(\mathbf{X}, \mathbf{V}),$$

$$P_{\rm cc}(\mathbf{X}, \mathbf{V}) = P_0(\mathbf{X}, \mathbf{V}) \cup P_{\rm v}(\mathbf{X}, \mathbf{V}),$$

where

$$P_{\rm cf}(\mathbf{X}, \mathbf{V}) \stackrel{\rm def}{=} \left\{ {\rm p} \in P_{\rm c}\colon \gamma_{\rm n}^{\rm p}(\mathbf{V}) < 0 \right\} \qquad \text{[particles in contact with negative normal velocity (\underline{f}ree in the near future)]}$$

$P_{cc}(\mathbf{X}, \mathbf{V}) \overset{\text{def}}{=} \{p \in P_c: \gamma_n^p(\mathbf{V}) = 0\}$ [particles in contact with no normal velocity (contact possibly preserved in the near future)]

$P_0(\mathbf{X}, \mathbf{V}) \overset{\text{def}}{=} \{p \in P_c: \gamma_n^p(\mathbf{V}) = \gamma_t^p(\mathbf{V}) = 0\}$ [particles in contact with normal and tangential velocities equal to 0]

$P_v(\mathbf{X}, \mathbf{V}) \overset{\text{def}}{=} \{p \in P_c: \gamma_n^p(\mathbf{V}) = \gamma_t^p(\mathbf{V}) \neq 0\}$ [particles in contact with vanishing normal velocity and non-vanishing tangential velocity].

**Remark 2.3.** Note that in situations at which no velocity discontinuity exists, the normal velocity $\gamma_n^p(\dot{\mathbf{X}}(t))$ is null at each particle p currently in contact, so that $P_{cf}(\mathbf{X}(t), \dot{\mathbf{X}}(t)) = \phi$. □

Another decomposition of the set of the particles that are currently in contact will also be useful in the following. For each $\mathbf{X} \in K_X$ and each $\mathbf{r} \in K_r(\mathbf{X})$, we let

$$P_c(\mathbf{X}) = P_z(\mathbf{X}, \mathbf{r}) \cup P_d(\mathbf{X}, \mathbf{r}) \cup P_s(\mathbf{X}, \mathbf{r}),$$

where the above disjoint subsets are

$P_z(\mathbf{X}, \mathbf{r}) \overset{\text{def}}{=} \{p \in P_c: r_n^p = r_t^p = 0\}$ [particles in contact with zero reaction]

$P_d(\mathbf{X}, \mathbf{r}) \overset{\text{def}}{=} \{p \in P_c: r_n^p < 0 \text{ and } |r_t^p| < -\mu r_n^p\}$ [particles in contact with reaction strictly inside the friction cone and consequent vanishing (right) displacement rate]

$P_s(\mathbf{X}, \mathbf{r}) \overset{\text{def}}{=} \{p \in P_c: r_n^p < 0 \text{ and } |r_t^p| = -\mu r_n^p\}$ [particles in contact with non-vanishing reaction on the boundary of the friction cone and consequent possible slip in the near future].

Observing that, for each particle currently in contact with no normal velocity [$p \in P_{cc}(\mathbf{X}(t), \dot{\mathbf{X}}(t))$, $d/dt(\phi^p(\mathbf{X}(t))) = 0$],

$$\frac{d^2}{dt^2}\phi^p(\mathbf{X}(t)) = \left|\frac{\partial \varphi^p}{\partial \mathbf{x}}(\mathbf{x}^p(\mathbf{X}(t)))\right| a_n^p(t) \tag{2.25}$$

$$a_n^p(t) \overset{\text{def}}{=} \frac{d}{dt}v_n^p(t) = \mathbf{G}_n^p(\mathbf{X}(t))\ddot{\mathbf{X}}(t) + \tilde{a}_n^p\big(\mathbf{X}(t), \dot{\mathbf{X}}(t)\big) \tag{2.26}$$

$$a_t^p(t) \overset{\text{def}}{=} \frac{d}{dt}v_t^p(t) = \mathbf{G}_t^p(\mathbf{X}(t))\ddot{\mathbf{X}}(t) + \tilde{a}_t^p\big(\mathbf{X}(t), \dot{\mathbf{X}}(t)\big), \tag{2.27}$$

where

$$\tilde{a}_n^p\big(\mathbf{X}, \dot{\mathbf{X}}\big) = \left[\mathbf{n}^p(\mathbf{x}^p(\mathbf{X})) \cdot \frac{\partial^2 \mathbf{x}^p}{\partial X_i \partial X_j}(\mathbf{X})\right]\dot{X}_i\dot{X}_j - \chi^p(\mathbf{x}^p(\mathbf{X}))\big(\mathbf{G}_t^p(\mathbf{X})\dot{\mathbf{X}}\big)^2 \tag{2.28}$$

$$\tilde{a}_t^p\big(\mathbf{X}, \dot{\mathbf{X}}\big) = \left[\mathbf{t}^p(\mathbf{x}^p(\mathbf{X})) \cdot \frac{\partial^2 \mathbf{x}^p}{\partial X_i \partial X_j}(\mathbf{X})\right]\dot{X}_i\dot{X}_j \tag{2.29}$$

and, having in mind $(2.15)_1$, (2.25) and (2.26), we define, for $\mathbf{X} \in K_X$ and $\mathbf{V} \in K_V(\mathbf{X})$, the (configuration and velocity dependent) *set of admissible generalized right accelerations*:

$$K_A(\mathbf{X}, \mathbf{V}) = \left\{ \mathbf{A} \in \mathbb{R}^N \colon \gamma_n(\mathbf{A}) + \tilde{a}_n \leqslant 0, \text{ in } P_{cc} \right\}. \tag{2.30}$$

On the other hand, for $\mathbf{X} \in K_X$, $\mathbf{V} \in K_V(\mathbf{X})$ and $\mathbf{A} \in K_A(\mathbf{X}, \mathbf{V})$, we define the (configuration, velocity and acceleration dependent) *set of admissible right reactions*:

$$\begin{aligned} K_r(\mathbf{X}, \mathbf{V}, \mathbf{A}) = \{\mathbf{r} \in \mathbb{R}^{2n_C} \colon & \\ & r_n \leqslant 0 \ \text{ and } \ \left(\gamma_n(\mathbf{A}) + \tilde{a}_n\right) r_n = 0, \text{ in } P_{cc}; \\ & r_n = 0, \text{ in } P_f \cup P_{cf}; \\ & r_t \in \mu r_n \sigma\left[\gamma_t(\mathbf{A}) + \tilde{a}_t\right], \text{ in } P_0; \\ & r_t = \mu r_n \sigma\left[\gamma_t(\mathbf{V})\right], \text{ in } P_v; \\ & r_t = 0, \text{ in } P_f \cup P_{cf}\} \\ \subset K_r(\mathbf{X}, \mathbf{V}) \subset K_r(\mathbf{X}). & \end{aligned} \tag{2.31}$$

Then the set $\mathbf{G}^T(\mathbf{X}) K_r(\mathbf{X}, \mathbf{V}, \mathbf{A})$ of *admissible generalized right reactions* can be characterized in the following manner.

**Lemma 2.4.** Let $\mathbf{X} \in K_X$, $\mathbf{V} \in K_V(\mathbf{X})$ and $\mathbf{A} \in K_A(\mathbf{X}, \mathbf{V})$ and assume that (2.9) holds for the matrix $\mathbf{G}(\mathbf{X})$. Then $\mathbf{R} \in \mathbf{G}^T(\mathbf{X}) K_r(\mathbf{X}, \mathbf{V}, \mathbf{A})$ if and only if

$$\mathbf{R} \cdot (\mathbf{A}' - \mathbf{A}) \geqslant \sum_{P_v} \mu \gamma_n^{\dagger}(\mathbf{R}) \sigma\left[\gamma_t(\mathbf{V})\right] \gamma_t(\mathbf{A}' - \mathbf{A}) + \sum_{P_0} \mu \gamma_n^{\dagger}(\mathbf{R}) \left(|\gamma_t(\mathbf{A}') + \tilde{a}_t| - |\gamma_t(\mathbf{A}) + \tilde{a}_t|\right), \quad \forall \mathbf{A}' \in K_A(\mathbf{X}, \mathbf{V}). \tag{2.32}$$

**Proof.** Let $\mathbf{R} \in \mathbf{G}^T(\mathbf{X}) K_r(\mathbf{X}, \mathbf{V}, \mathbf{A})$. Then, by (2.10),

$$\begin{aligned} \mathbf{R} \cdot (\mathbf{A}' - \mathbf{A}) &= \mathbf{G}^T \mathbf{G}^{\dagger} \mathbf{R} \cdot (\mathbf{A}' - \mathbf{A}) \\ &= \mathbf{G}^{\dagger} \mathbf{R} \cdot \mathbf{G}(\mathbf{A}' - \mathbf{A}) \\ &= \mathbf{G}^{\dagger} \mathbf{R} \cdot \left[ (\mathbf{G}\mathbf{A}' + \tilde{\mathbf{a}}) - (\mathbf{G}\mathbf{A} + \tilde{\mathbf{a}}) \right] \end{aligned}$$

with $\mathbf{G}^{\dagger}\mathbf{R} \in K_r(\mathbf{X}, \mathbf{V}, \mathbf{A})$. But in $P_{cc}$ the following holds: $\gamma_n(\mathbf{A}) + \tilde{a}_n \leqslant 0$, and $\gamma_n(\mathbf{A}') + \tilde{a}_n \leqslant 0$, for all $\mathbf{A}' \in K_A(\mathbf{X}, \mathbf{V})$. Then the inequality (2.32) follows from Lemma 2.6 in Martins et al. (1988).

Let now the inequality statement (2.32) hold and let $\mathbf{A}' - \mathbf{A} = \pm \boldsymbol{\alpha}$ be an arbitrary element of *Ker* ($\mathbf{G}$). Then (2.32) implies that $\pm \mathbf{R} \cdot \boldsymbol{\alpha} \geqslant 0$ for all $\boldsymbol{\alpha} \in Ker$ ($\mathbf{G}$), hence $\mathbf{R} \in [Ker(\mathbf{G})]^{\perp} = Rg(\mathbf{G}^T)$ and, proceeding as above, the statement (2.32) reads

$$\mathbf{G}^{\dagger}\mathbf{R}\cdot\left[(\mathbf{GA}'+\tilde{\mathbf{a}})-(\mathbf{GA}+\tilde{\mathbf{a}})\right]\geqslant\sum_{P_{\mathrm{v}}}\mu\gamma_{\mathrm{n}}^{\dagger}(\mathbf{R})\sigma\left[\gamma_{\mathrm{t}}(\mathbf{V})\right]\left[\left(\gamma_{\mathrm{t}}(\mathbf{A}')+\tilde{a}_{\mathrm{t}}\right)-\left(\gamma_{\mathrm{t}}(\mathbf{A})+\tilde{a}_{\mathrm{t}}\right)\right]$$

$$+\sum_{P_0}\mu\gamma_{\mathrm{n}}^{\dagger}(\mathbf{R})\left(|\gamma_{\mathrm{t}}(\mathbf{A}')+\tilde{a}_{\mathrm{t}}|-|\gamma_{\mathrm{t}}(\mathbf{A})+\tilde{a}_{\mathrm{t}}|\right),\quad\forall\mathbf{A}'\in K_{\mathrm{A}}(\mathbf{X},\mathbf{V}).$$

The desired result, $\mathbf{G}^{\dagger}\mathbf{R}\in K_{\mathrm{r}}(\mathbf{X},\mathbf{V},\mathbf{A})$, follows then from Lemma 2.6 in Martins et al. (1998) and from the surjectivity (2.9) of the linear map $\mathbf{G}$: $\mathbb{R}^N\to\mathbb{R}^{2n_{\mathrm{C}}}$. □

The admissible right accelerations and reactions can be characterized in the following alternative manner. For $\mathbf{X}\in K_{\mathrm{X}}$, $\mathbf{V}\in K_{\mathrm{V}}(\mathbf{X})$ and $\mathbf{r}\in K_{\mathrm{r}}(\mathbf{X},\mathbf{V})$, we define the (configuration, reaction and velocity dependent) *set of admissible right accelerations*

$$\begin{aligned}K_{\mathrm{A}}(\mathbf{X},\mathbf{r},\mathbf{V})\stackrel{\mathrm{def}}{=}\{&\mathbf{A}\in\mathbb{R}^N:\\&\gamma_{\mathrm{n}}(\mathbf{A})+\tilde{a}_{\mathrm{n}}\leqslant 0\ \text{and}\ \left(\gamma_{\mathrm{n}}(\mathbf{A})+\tilde{a}_{\mathrm{n}}\right)r_{\mathrm{n}}=0,\ \text{in}\ P_{\mathrm{cc}};\\&\gamma_{\mathrm{t}}(\mathbf{A})+\tilde{a}_{\mathrm{t}}=0,\ \text{in}\ P_0\cap P_{\mathrm{d}};\\&\left(\gamma_{\mathrm{t}}(\mathbf{A})+\tilde{a}_{\mathrm{t}}\right)\sigma[r_{\mathrm{t}}]\leqslant 0,\ \text{in}\ P_0\cap P_{\mathrm{s}}\}\end{aligned}\tag{2.33}$$

Then this set admits the following variational characterization, the proof of which is given in Appendix A.

**Lemma 2.5.** Let $\mathbf{X}\in K_{\mathrm{X}}$, $\mathbf{V}\in K_{\mathrm{V}}(\mathbf{X})$ and $\mathbf{r}\in K_{\mathrm{r}}(\mathbf{X},\mathbf{V})$. Then $\mathbf{A}\in K_{\mathrm{A}}(\mathbf{X},\mathbf{r},\mathbf{V})$ if and only if

$$(\mathbf{GA}+\tilde{\mathbf{a}})\cdot(\mathbf{r}'-\mathbf{r})\geqslant\sum_{P_{\mathrm{v}}}\mu\sigma\left[\gamma_{\mathrm{t}}(\mathbf{V})\right]\left(\gamma_{\mathrm{t}}(\mathbf{A})+\tilde{a}_{\mathrm{t}}\right)\left(r_{\mathrm{n}}'-r_{\mathrm{n}}\right)+\sum_{P_0}\mu|\gamma_{\mathrm{t}}(\mathbf{A})+\tilde{a}_{\mathrm{t}}|\left(r_{\mathrm{n}}'-r_{\mathrm{n}}\right),$$
$$\forall\mathbf{r}'\in K_{\mathrm{r}}(\mathbf{K},\mathbf{V}).\tag{2.34}$$ □

*2.3. Admissible right velocities and reaction rates*

For $\mathbf{X}\in K_{\mathrm{X}}$ and $\mathbf{r}\in K_{\mathrm{r}}(\mathbf{X})$ we define the (configuration and reaction dependent) *set of admissible right generalized velocities*,

$$\begin{aligned}K_{\mathrm{V}}(\mathbf{X},\mathbf{r})\stackrel{\mathrm{def}}{=}\{&\mathbf{V}\in\mathbb{R}^N:\\&\gamma_{\mathrm{n}}(\mathbf{V})\leqslant 0,\ \text{in}\ P_{\mathrm{z}};\\&\gamma_{\mathrm{n}}(\mathbf{V})=0,\ \text{in}\ P_{\mathrm{s}}\cup P_{\mathrm{d}};\\&\sigma[r_{\mathrm{t}}]\gamma_{\mathrm{t}}(\mathbf{V})\leqslant 0,\ \text{in}\ P_{\mathrm{s}};\\&\gamma_{\mathrm{t}}(\mathbf{V})=0,\ \text{in}\ P_{\mathrm{d}}\}\\\subset\ &K_{\mathrm{V}}(\mathbf{X}).\end{aligned}\tag{2.35}$$

Having in mind (2.24), the (configuration, reaction and velocity dependent) *set of admissible right reaction rates* is defined by

$$\begin{aligned}K_{\mathbf{w}}(\mathbf{X},\mathbf{r},\mathbf{V})\stackrel{\mathrm{def}}{=}\{&\mathbf{w}\in\mathbb{R}^{2n_{\mathrm{C}}}:\\&w_{\mathrm{n}}\leqslant 0,\ w_{\mathrm{n}}\gamma_{\mathrm{n}}(\mathbf{V})=0,\ \text{in}\ P_{\mathrm{z}};\\&w_{\mathrm{n}}=0,\ \text{in}\ P_{\mathrm{f}};\end{aligned}$$

$$\sigma[r_t]w_t + \mu w_n \leqslant 0 \text{ and } (w_t\sigma[r_t] + \mu w_n)\big(\sigma[r_t]\gamma_t(\mathbf{V})\big) = 0, \text{ in } P_s;$$
$$w_t \in \mu w_n \sigma\big[\gamma_t(\mathbf{V})\big], \text{ in } P_z;$$
$$w_t = 0, \text{ in } P_f\}. \tag{2.36}$$

Then the set $\mathbf{G}^T(\mathbf{X})K_\mathbf{w}(\mathbf{X}, \mathbf{r}, \mathbf{V})$ of generalized forces corresponding to the right reaction rates admits the following characterization.

**Lemma 2.6.** Let $\mathbf{X} \in K_X$, $\mathbf{r} \in K_r(\mathbf{X})$ and $\mathbf{V} \in K_V(\mathbf{X}, \mathbf{r})$. Then $\mathbf{W} \in \mathbf{G}^T(\mathbf{X})K_\mathbf{w}(\mathbf{X}, \mathbf{r}, \mathbf{V})$ if and only if

$$\begin{aligned}\mathbf{W} \cdot (\mathbf{V}' - \mathbf{V}) &\geqslant \sum_{P_c} \mu\gamma_n^\dagger(\mathbf{W})\big(|\gamma_t(\mathbf{V}')| - |\gamma_t(\mathbf{V})|\big)\\ &= \sum_{P_z} \mu\gamma_n^\dagger(\mathbf{W})\big(|\gamma_t(\mathbf{V}')| - |\gamma_t(\mathbf{V})|\big) - \sum_{P_s} \mu\gamma_n^\dagger(\mathbf{W})\sigma[r_t]\big(\gamma_t(\mathbf{V}') - \gamma_t(\mathbf{V})\big),\\ &\qquad \forall \mathbf{V}' \in K_V(\mathbf{X}, \mathbf{r}).\end{aligned} \tag{2.37}$$

**Proof.** This result follows from Lemma 2.7 in Martins et al. (1998) by using arguments similar to those used in Lemma 2.4 above. □

The admissible right velocities and reaction rates can be characterized in the following alternative manner. For each $\mathbf{X} \in K_X$ and each $\mathbf{r} \in K_r(\mathbf{X})$, we define the (configuration and reaction dependent) *set of admissible right reaction rates*

$$\begin{aligned}K_\mathbf{w}(\mathbf{X}, \mathbf{r}) \stackrel{\text{def}}{=} \{\mathbf{w} \in \mathbb{R}^{2n_C}:&\\ & w_n \leqslant 0, \text{ in } P_z;\\ & w_n = 0, \text{ in } P_f;\\ & |w_t| + \mu w_n \leqslant 0, \text{ in } P_z;\\ & \sigma[r_t]w_t + \mu w_n \leqslant 0, \text{ in } P_s;\\ & w_t = 0, \text{ in } P_f\}\\ = K_\mathbf{w}(\mathbf{X}, \mathbf{r}, \mathbf{0}).&\end{aligned} \tag{2.38}$$

Having then $\mathbf{w} \in K_\mathbf{w}(\mathbf{X}, \mathbf{r})$, we define the (configuration, reaction and reaction rate dependent) *set of admissible right generalized velocities*,

$$\begin{aligned}K_V(\mathbf{X}, \mathbf{r}, \mathbf{w}) \stackrel{\text{def}}{=} \{\mathbf{V} \in \mathbb{R}^N:&\\ & \gamma_n(\mathbf{V}) \leqslant 0,\ w_n\gamma_n(\mathbf{V}) = 0, \text{ in } P_z;\\ & \gamma_n(\mathbf{V}) = 0, \text{ in } P_s \cup P_d;\\ & w_t\gamma_t(\mathbf{V}) \leqslant 0 \text{ and } [|w_t| + \mu w_n]\gamma_t(\mathbf{V}) = 0, \text{ in } P_z;\\ & \sigma[r_t]\gamma_t(\mathbf{V}) \leqslant 0 \text{ and } (\sigma[r_t]w_t + \mu w_n)\big(\sigma[r_t]\gamma_t(\mathbf{V})\big) = 0, \text{ in } P_s;\\ & \gamma_t(\mathbf{V}) = 0, \text{ in } P_d\}.\end{aligned} \tag{2.39}$$

This set can be characterized by the variational inequality given in the next Lemma, which is established essentially in the same manner as the one of Lemma 2.5.

**Lemma 2.7.** Let $\mathbf{X} \in K_X$, $\mathbf{r} \in K_r(\mathbf{X})$ and $\mathbf{w} \in K_\mathbf{w}(\mathbf{X}, \mathbf{r})$. Then $\mathbf{V} \in K_V(\mathbf{X}, \mathbf{r}, \mathbf{w})$ if and only if

$$\mathbf{GV} \cdot (\mathbf{w}' - \mathbf{w}) \geqslant \sum_{P_c} \mu |\gamma_t(\mathbf{V})| \left(w'_n - w_n\right)$$
$$= \sum_{P_z} \mu |\gamma_t(\mathbf{V})| \left(w'_n - w_n\right) - \sum_{P_s} \mu \sigma[r_t] \gamma_t(\mathbf{V}) \left(w'_n - w_n\right), \quad \forall \mathbf{w}' \in K_\mathbf{w}(\mathbf{X}, \mathbf{r}). \qquad (2.40) \quad \square$$

## 3. Initial acceleration and reaction discontinuities

The initial conditions considered in this section are

$$\mathbf{X}(\tau) = \mathbf{X}^0 \in K_X \qquad (3.1)$$

$$\dot{\mathbf{X}}(\tau) = \mathbf{0}, \qquad (3.2)$$

where the equilibrium configuration $\mathbf{X}^0$ corresponds to some equilibrium reactions $\mathbf{r}^0 \in K_r(\mathbf{X}^0)$ such that [recall (2.13), (2.14) and (2.18)] $\mathbf{B}(\mathbf{X}^0, \mathbf{0}) = -\mathbf{F}^U(\mathbf{X}^0) - \mathbf{F}^\Omega(\mathbf{X}^0) = \mathbf{G}^T(\mathbf{X}^0)\mathbf{r}^0$. We begin by proving in Section 3.1 some necessary conditions for the existence of dynamic solutions initiating at the equilibrium configuration, with no initial perturbation, but with an acceleration and reaction discontinuity at the initial time $\tau$. In Section 3.2 a sufficient condition is presented which guarantees that, in some circumstances, a smooth portion of a non-trivial dynamic solution actually follows that initial discontinuity.

### 3.1. Necessary conditions for the occurrence of initial acceleration and reaction discontinuities

For each $\mathbf{X} \in K_X$ and each pair $(\mathbf{A}, \mathbf{A}') \in \mathbb{R}^N \times \mathbb{R}^N$ we define

$$m^*\left(\mathbf{A}, \mathbf{A}'\right) \stackrel{\text{def}}{=} \mathbf{MA} \cdot \mathbf{A}' - \sum_{P_c} \mu \gamma_n^\dagger(\mathbf{MA}) \left|\gamma_t(\mathbf{A}')\right|. \qquad (3.3)$$

On the other hand, for each $\mathbf{X} \in K_X$ and each pair $(\mathbf{r}, \mathbf{r}') \in \mathbb{R}^{2nc} \times \mathbb{R}^{2nc}$, we define

$$m^\#\left(\mathbf{r}, \mathbf{r}'\right) = \mathbf{GM}^{-1}\mathbf{G}^T\mathbf{r} \cdot \mathbf{r}' - \sum_{P_c} \mu \left|\gamma_t(\mathbf{M}^{-1}\mathbf{G}^T\mathbf{r})\right| r'_n. \qquad (3.4)$$

Then we have the following results.

**Proposition 3.1.** Assume that (2.9) holds. Then a dynamic solution with initial conditions (3.1) and (3.2) may be initiated with an acceleration discontinuity only if

$\exists \mathbf{A} \in K_A\left(\mathbf{X}^0, \mathbf{0}\right)$, $\mathbf{A} \neq \mathbf{0}$, such that

$$\mathbf{MA} + \mathbf{G}^T\mathbf{r}^0 \in \mathbf{G}^T K_r\left(\mathbf{X}^0, \mathbf{0}, \mathbf{A}\right), \qquad (3.5)$$

i.e. such that

$$m^*(\mathbf{A}, \mathbf{A}') - m^*(\mathbf{A}, \mathbf{A}) + \left[ \mathbf{r}^0 \cdot \mathbf{G}(\mathbf{A}' - \mathbf{A}) - \sum_{P_c} \mu r_n^0 \big( |\gamma_t(\mathbf{A}')| - |\gamma_t(\mathbf{A})| \big) \right] \geqslant 0, \quad \forall \mathbf{A}' \in K_A(\mathbf{X}^0, \mathbf{0}). \tag{3.6}$$

**Proof.** Let $\mathbf{A} \in K_A(\mathbf{X}^0, \mathbf{0})$, with $\mathbf{A} \neq \mathbf{0}$, be the acceleration discontinuity. Then there must exist $\mathbf{r}^+ \in K_r(\mathbf{X}^0, \mathbf{0}, \mathbf{A})$ such that

$$\mathbf{MA} = \mathbf{G}^T(\mathbf{r}^+ - \mathbf{r}^0), \tag{3.7}$$

i.e. (3.5) holds. Then inequality (3.6) follows from (2.32) and the definition (3.3) of $m^*$. □

**Corollary 3.2.** Let (2.9) hold.

(i) If (3.4) holds then:

$$m^*(\mathbf{A}, \mathbf{A}) \leqslant 0. \tag{3.8}$$

(ii) No dynamic solution with initial conditions (3.1) and (3.2) may be initiated with an acceleration discontinuity if

$$m^*(\mathbf{A}', \mathbf{A}') > 0, \quad \forall \mathbf{A}' \in K_A(\mathbf{X}^0, \mathbf{0}), \mathbf{A}' \neq \mathbf{0}. \tag{3.9}$$

**Proof.** Since, as already mentioned, $\mathbf{r}^0 \in K_r(\mathbf{X}^0) = K_r(\mathbf{X}^0, \mathbf{0}, \mathbf{0})$, it follows for Lemma 2.4 that

$$\mathbf{r}^0 \cdot \mathbf{GA}' \geqslant \sum_{P_c} \mu r_n^0 |\gamma_t(\mathbf{A}')|, \quad \forall \mathbf{A}' \in K_A(\mathbf{X}^0, \mathbf{0}). \tag{3.10}$$

Then, taking $\mathbf{A}' = \mathbf{A}$ in (3.10) and taking successively $\mathbf{A}' = \mathbf{0}$ and $\mathbf{A}' = 2\mathbf{A}$ in (3.6) we get

$$-\mathbf{r}^0 \cdot \mathbf{GA} + \sum_{P_c} \mu r_n^0 |\gamma_t(\mathbf{A})| \leqslant 0,$$

$$m^*(\mathbf{A}, \mathbf{A}) = -\mathbf{r}^0 \cdot \mathbf{GA} + \sum_{P_c} \mu r_n^0 |\gamma_t(\mathbf{A})|,$$

which imply the result (3.8). The result (3.9) follows by negation of (3.8). □

**Proposition 3.3.** A dynamic solution with initial conditions (3.1) and (3.2) may be initiated with an acceleration discontinuity only if

$$\exists \Delta\mathbf{r} \in K_r(\mathbf{X}^0, \mathbf{0}) - \mathbf{r}^0, \quad \Delta\mathbf{r} \neq \mathbf{0}, \text{ such that}$$

$$\mathbf{M}^{-1}\mathbf{G}^T \Delta\mathbf{r} \in K_A(\mathbf{X}^0, \mathbf{r}^0 + \Delta\mathbf{r}, \mathbf{0}) \tag{3.11}$$

i.e. such that

$$m^{\#}\big(\Delta\mathbf{r}, \Delta\mathbf{r}' - \Delta\mathbf{r}\big) \geqslant 0, \quad \forall \Delta\mathbf{r}' \in K_{\mathrm{r}}\big(\mathbf{X}^0, \mathbf{0}\big) - \mathbf{r}^0. \tag{3.12}$$

**Proof.** The right reactions and accelerations satisfy $\mathbf{r}^+ \in K_{\mathrm{r}}(\mathbf{X}^0, \mathbf{0})$ and $\mathbf{A} \in K_{\mathrm{A}}(\mathbf{X}^0, \mathbf{r}^+, \mathbf{0})$. Then, by Lemma 2.5 we obtain

$$\mathbf{GA} \cdot (\mathbf{r}' - \mathbf{r}^+) \geqslant \sum_{P_c} \mu \left|\gamma_{\mathrm{t}}(\mathbf{A})\right| \left(r'_{\mathrm{n}} - r^+_{\mathrm{n}}\right), \quad \forall \mathbf{r}' \in K_{\mathrm{r}}\big(\mathbf{X}^0, \mathbf{0}\big).$$

However, $\mathbf{A} = \mathbf{M}^{-1}\mathbf{G}^{\mathrm{T}}(\mathbf{r}^+ - \mathbf{r}^0) \neq \mathbf{0}$, so that $\mathbf{r}^+ - \mathbf{r}^0 \neq \mathbf{0}$. The result follows then by taking the arbitrary $\mathbf{r}' \in K_{\mathrm{r}}(\mathbf{X}^0, \mathbf{0})$ in the form $\mathbf{r}' = \mathbf{r}^0 + \Delta\mathbf{r}'$ with $\Delta\mathbf{r}' \in K_{\mathrm{r}}(\mathbf{X}^0,\mathbf{0}) - \mathbf{r}^0$ arbitrary, and by using the definition (3.4) of $m^{\#}$. □

**Corollary 3.4.** (i) If (3.11) holds then:

$$m^{\#}(\Delta\mathbf{r}, \Delta\mathbf{r}) \leqslant 0. \tag{3.13}$$

(ii) No dynamic solution with initial conditions (3.1) and (3.2) may be initiated with an acceleration discontinuity if

$$m^{\#}\big(\mathbf{w}', \mathbf{w}'\big) > 0, \quad \forall \mathbf{w}' \in K_{\mathbf{w}}\big(\mathbf{X}^0, \mathbf{r}^0\big), \quad \mathbf{w}' \neq \mathbf{0}. \tag{3.14}$$

**Proof.** The inequality (3.13) follows from (3.12) by taking the arbitrary $\Delta\mathbf{r}'$ in (3.12) to be null. On the other hand, (3.14) follows by negation of (3.13) and by observing that $K_{\mathrm{r}}(\mathbf{X}^0, \mathbf{0}) - \mathbf{r}^0 \subset K_{\mathbf{w}}(\mathbf{X}^0, \mathbf{r}^0)$. □

**Remark 3.5.** Note that, if (2.9) holds, the necessary conditions (3.5) and (3.11) for the occurrence of an acceleration discontinuity are *equivalent*. This happens because $(\mathbf{A}, \mathbf{r}^0 + \Delta\mathbf{r}) \in K_{\mathrm{A}}(\mathbf{X}^0, \mathbf{0}) \times K_{\mathrm{r}}(\mathbf{X}^0, \mathbf{0}, \mathbf{A})$ is equivalent to $(\mathbf{A}, \mathbf{r}^0 + \Delta\mathbf{r}) \in K_{\mathrm{A}}(\mathbf{X}^0, \mathbf{r}^0 + \Delta\mathbf{r}, \mathbf{0}) \times K_{\mathrm{r}}(\mathbf{X}^0, \mathbf{0})$, and because (2.9) implies *Ker* $(\mathbf{G}^{\mathrm{T}}) = \{\mathbf{0}\}$. Consequently, from $\mathbf{G}^{\mathrm{T}}\Delta\mathbf{r} = \mathbf{MA}$, not only it follows (as used in Proposition 3.3) that $\mathbf{A} \neq \mathbf{0} \Longrightarrow \Delta\mathbf{r} \neq \mathbf{0}$, but also that $\Delta\mathbf{r} \neq \mathbf{0} \Longrightarrow \mathbf{A} \neq \mathbf{0}$. □

Propositions 3.1 and 3.3 give nothing but *necessary* conditions for the occurrence of initial acceleration (and reaction) discontinuities. A *sufficient* condition must guarantee that these discontinuities can actually be followed by (smooth) non-trivial dynamic solutions starting from the equilibrium configuration. One such sufficient condition will be established next.

We assume that (2.9) holds and that for some $\mathbf{A}$ and $\Delta\mathbf{r}$ satisfying (3.5) and (31) with $\mathbf{G}^{\mathrm{T}}\Delta\mathbf{r}{=}\mathbf{MA} \neq \mathbf{0}$, the following holds for all particles in $P_{\mathrm{c}}(\mathbf{X}^0)$:

either:

$$\gamma_{\mathrm{n}}(\mathbf{A}) < 0 \text{ and } r^0_{\mathrm{n}} + \Delta r_{\mathrm{n}} = r^0_{\mathrm{t}} + \Delta r_{\mathrm{t}} = 0, \quad \big[\text{near future } \textit{free}\big] \tag{3.15}$$

or:

$$\gamma_{\mathrm{n}}(\mathbf{A}) = 0, \quad r^0_{\mathrm{n}} + \Delta \mathrm{r}_{\mathrm{n}} < 0$$

and

$$\gamma_{\mathrm{t}}(\mathbf{A}) \neq 0 \text{ and } \big(r^0_{\mathrm{t}} + \Delta r_{\mathrm{t}}\big)\sigma\big[\gamma_{\mathrm{t}}(\mathbf{A})\big] + \mu\big(r^0_{\mathrm{n}} + \Delta r_{\mathrm{n}}\big) = 0 \quad \big[\text{near future } \textit{slip}\big] \tag{3.16}$$

or

$$\gamma_t(\mathbf{A}) = 0 \text{ and } |r_t^0 + \Delta r_t| < -\mu(r_n^0 + \Delta r_n) \quad [\text{near future } stick]. \tag{3.17}$$

This means that the acceleration and reaction jump vectors **A** and $\Delta\mathbf{r}$ give an unambiguous information on the near future states of each particle in contact at the equilibrium configuration: strictly *free* states if (3.15) holds, strictly *stick* states if (3.16) holds, and strictly *slip* states if (3.17) holds. *Assuming* that such states hold in some right neighborhood of $\tau$, we consider the following decompositions of the reaction vector **r** and of the kinematic matrix **G**:

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_{free} \\ \mathbf{r}_{t\,slip} \\ \hat{\mathbf{r}} \end{bmatrix}, \quad \hat{\mathbf{r}} = \begin{bmatrix} \mathbf{r}_{n\,slip} \\ \mathbf{r}_{stick} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}_{free} \\ \mathbf{G}_{t\,slip} \\ \hat{\mathbf{G}} \end{bmatrix}, \quad \hat{\mathbf{G}} = \begin{bmatrix} \mathbf{G}_{nslip} \\ \mathbf{G}_{stick} \end{bmatrix}. \tag{3.18}$$

Note that the set of the *free* particles contains those that are already free at the equilibrium configuration $[\phi(\mathbf{X}^0) < 0]$ and those that become free in the near future (3.15). Then the equations of motion (2.13) and the unilateral friction contact conditions (2.15) and (2.16) yield the system of differential equations:

$$\mathbf{M}(\mathbf{X}(t))\ddot{\mathbf{X}}(t) + \mathbf{B}\big(\mathbf{X}(t), \dot{\mathbf{X}}(t)\big) = \hat{\mathbf{G}}_{\mu}^{T}(\mathbf{X}(t))\hat{\mathbf{r}}(t), \tag{3.19}$$

together with the equality constraints:

$$\hat{\mathbf{G}}(\mathbf{X}(t))\dot{\mathbf{X}}(t) = \mathbf{0}, \tag{3.20}$$

where

$$\hat{\mathbf{G}}_{\mu}^{T} = \Big[\mathbf{G}_{n\,slip}^{T} + \mu\mathbf{G}_{t\,slip}^{T}\mathbf{S}_{slip} \quad \mathbf{G}_{stick}^{T}\Big]. \tag{3.21}$$

Note that in the construction of the problem (3.19) and (3.20) the kinematic equality constraints indicated in column I of Table 1 are taken into account by means of (3.20) and the equality constraints involving the reaction forces (see column II of Table 1) are taken into account in the right-hand side of

Table 1
Contact conditions satisfied by the nonlinear system in a sufficiently small neighborhood of the equilibrium after an acceleration-reaction discontinuity

| | Equality conditions imposed on the construction of the nonlinear equations of motion | | Remaining inequality conditions verified by the solution while it is valid | |
|---|---|---|---|---|
| | I kinematic | II static | III kinematic | IV static |
| Free | | $r_n = 0$<br>$r_t = 0$ | $\phi(\mathbf{X}) < 0$ | |
| Stick | $\phi(\mathbf{X}) = 0$<br>$\gamma_t(\dot{\mathbf{X}}) = 0$ | | | $r_n < 0$<br>$\lvert r_t\rvert + \mu r_n < 0$ |
| Slip | $\phi(\mathbf{X}) = 0$ | $r_t - \sigma[\gamma_t(\mathbf{A})]\mu r_n = 0$ | $\sigma[\gamma_t(\mathbf{A})]\gamma_t(\dot{\mathbf{X}}) > 0$ | $r_n < 0$ |

(3.19). In particular,

$$\mathbf{r}_{\mathrm{t}\ slip} = \mu \mathbf{S}_{slip} \mathbf{r}_{\mathrm{n}\ slip}; \quad \mathbf{S}_{slip} = \mathrm{diag}\ (\sigma_{\mathrm{p}}); \quad \sigma_{\mathrm{p}} = \sigma\big[\gamma_{\mathrm{t}}^{\mathrm{p}}(\mathbf{A})\big], \text{ for each } slip \text{ particle p.}$$

Note also that the solution $(\mathbf{X}(t), \hat{\mathbf{r}}(t))$ to the problem (3.19) and (3.20) will be a solution to the original problem (2.13), (2.15) and (2.16) in a time interval $[\tau, \tau + \Delta\tau[$ if the strict inequalities indicated in columns III and IV of Table 1 are valid in $]\tau, \tau + \Delta\tau[$ and if the accelerations $\ddot{\mathbf{X}}$ and the reactions $\hat{\mathbf{r}}$ are continuous functions of time in $[\tau, \tau + \Delta\tau[$.

We now proceed to eliminate the reactions $\hat{\mathbf{r}}$ from eqns (3.19). This can be achieved in two ways.

The first procedure starts by using the *assumption* that $\mathbf{G}$ (and consequently $\hat{\mathbf{G}}$) *is a full rank matrix* (2.9). In these circumstances if follows from (2.13) that

$$\hat{\mathbf{r}} = \hat{\mathbf{I}} \mathbf{G}^{\dagger} \big(\mathbf{M}\ddot{\mathbf{X}} + \mathbf{B}\big), \tag{3.22}$$

where, using the decomposition (3.18) of the reactions $\mathbf{r}$ into $\mathbf{r}_{free}$, $\mathbf{r}_{\mathrm{t}\ slip}$ and $\hat{\mathbf{r}}$,

$$\hat{\mathbf{I}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

is an $\hat{n} \times 2n_{\mathrm{C}}$ matrix; $\hat{n} = n_{slip} + 2n_{stick}$, $n_{slip}$ and $n_{stick}$ being the number of *slip* and *stick* particles, respectively; note that in this paper all identity matrices are denoted simply by $\mathbf{I}$, independently of the appropriate dimensions they have in each case. In this manner (3.19) becomes

$$\Big[\mathbf{I} - \hat{\mathbf{G}}_{\mu}^{\mathrm{T}} \hat{\mathbf{I}} \mathbf{G}^{\dagger}\Big]\big[\mathbf{M}\ddot{\mathbf{X}} + \boldsymbol{B}\big] = \mathbf{0}. \tag{3.23}$$

The kinematic constraints (3.20) are taken into account by selecting a subvector of $\dot{\mathbf{X}}$ containing only independent velocities. In fact, since $\hat{\mathbf{G}}$ is a full rank matrix, a square non-singular submatrix $\hat{\mathbf{G}}^{\mathrm{D}}$ of $\hat{\mathbf{G}}$ exists such that

$$\hat{\mathbf{G}}\dot{\mathbf{X}} = \hat{\mathbf{G}}^{*}\dot{\mathbf{X}}^{*} + \hat{\mathbf{G}}^{\mathrm{D}}\dot{\mathbf{X}}^{\mathrm{D}} = \mathbf{0}, \tag{3.24}$$

where $\dot{\mathbf{X}}^{*}$ is an $N^{*} \times 1$ vector of independent velocities $(N^{*} = N - \hat{n})$ and $\dot{\mathbf{X}}^{\mathrm{D}}$ is an $\hat{n} \times 1$ vector of dependent velocities (Blajer et al., 1994; Wehage and Haug, 1982). In general this partition of the vector $\dot{\mathbf{X}}$ is not unique. A criterion to select a best partition is given by Blajer et al. (1994). Using (3.24) to eliminate the dependent velocities $\dot{\mathbf{X}}^{\mathrm{D}}$, the vector $\dot{\mathbf{X}}$ is related to $\dot{\mathbf{X}}^{*}$ by

$$\dot{\mathbf{X}} = \mathbf{C}^{\mathrm{T}}(\mathbf{X})\dot{\mathbf{X}}^{*}, \tag{3.25}$$

where the $N^{*} \times N$ matrix $C$ is given by

$$\mathbf{C} = \Big[\mathbf{I} \quad -\Big(\big(\hat{\mathbf{G}}^{\mathrm{D}}\big)^{-1}\hat{\mathbf{G}}^{*}\Big)^{\mathrm{T}}\Big]. \tag{3.26}$$

Introducing now (3.25) in (3.19) and (Blajer et al., 1994) using the operator $\mathbf{C}$ to project the dynamic eqns (3.23) on the directions of the configuration space that are tangential to the constraints, the reduced system of nonlinear equations

$$\mathbf{M}^{*}\ddot{\mathbf{X}}^{*} + \mathbf{B}^{*} = \mathbf{0} \tag{3.27}$$

is obtained, where

$$\mathbf{M}^* = \mathbf{M}^*(\mathbf{X}) = \mathbf{C}\Big[\mathbf{I} - \hat{\mathbf{G}}_\mu^{\mathrm{T}}\hat{\mathbf{I}}\mathbf{G}^\dagger\Big]\mathbf{M}\mathbf{C}^{\mathrm{T}} \tag{3.28}$$

and

$$\mathbf{B}^* = \mathbf{B}^*\big(\mathbf{X}, \dot{\mathbf{X}}^*\big) = \mathbf{C}\Big[\mathbf{I} - \hat{\mathbf{G}}_\mu^{\mathrm{T}}\hat{\mathbf{I}}\mathbf{G}^\dagger\Big]\big(\mathbf{B} + \mathbf{M}\dot{\mathbf{C}}^{\mathrm{T}}\dot{\mathbf{X}}^*\big) \tag{3.29}$$

are the effective mass matrix and the effective vector of nonlinear terms, both affected by the coefficient of friction. In the sequel it will be useful to observe that

$$\hat{\mathbf{G}}_\mu^{\mathrm{T}} = \mathbf{G}^{\mathrm{T}}\mathbf{H}_\mu$$

where, using the decompositions (3.18) of $\mathbf{r}$ into $\mathbf{r}_{free}$, $\mathbf{r}_{\mathrm{t}\,slip}$, $\mathbf{r}_{\mathrm{n}\,slip}$, and $\mathbf{r}_{stick}$, and of $\hat{\mathbf{r}}$ into $\mathbf{r}_{\mathrm{n}\,slip}$ and $\mathbf{r}_{stick}$, the $2n_{\mathrm{C}} \times \hat{n}$ matrix $\mathbf{H}_\mu$ is defined by

$$\mathbf{H}_\mu = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mu\mathbf{S}_{slip} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

The other procedure for the elimination of $\hat{\mathbf{r}}$ in (3.19) starts by projecting those equations on the directions of the configuration space that are normal to the constraints, which corresponds to premultiplying (3.19) by $\hat{\mathbf{G}}\mathbf{M}^{-1}$ (Lötstedt, 1982a, b; Blajer et al., 1994; Blajer, 1995; Blajer and Markiewicz, 1995). We get

$$\hat{\mathbf{G}}\ddot{\mathbf{X}} + \hat{\mathbf{G}}\mathbf{M}^{-1}\mathbf{B} = \hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_\mu^{\mathrm{T}}\hat{\mathbf{r}},$$

where, differentiating the constraints (3.20) with respect to time,

$$\hat{\mathbf{G}}\ddot{\mathbf{X}} = -\dot{\hat{\mathbf{G}}}\dot{\mathbf{X}}.$$

Then, if the matrix $\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_\mu^{\mathrm{T}}$ *is invertible*, we get

$$\hat{\mathbf{r}} = \Big(\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_\mu^{\mathrm{T}}\Big)^{-1}\Big(\hat{\mathbf{G}}\mathbf{M}^{-1}\mathbf{B} - \dot{\hat{\mathbf{G}}}\dot{\mathbf{X}}\Big), \tag{3.30}$$

which is substituted in the right-hand side of (3.19) yielding:

$$\mathbf{M}\ddot{\mathbf{X}} + \bar{\mathbf{B}} = \mathbf{0}, \tag{3.31}$$

where

$$\bar{\mathbf{B}} = \bar{\mathbf{B}}\big(\mathbf{X}, \dot{\mathbf{X}}\big) = \Big(\mathbf{I} - \hat{\mathbf{G}}_\mu^{\mathrm{T}}\Big(\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_\mu^{\mathrm{T}}\Big)^{-1}\hat{\mathbf{G}}\mathbf{M}^{-1}\Big)\mathbf{B} + \hat{\mathbf{G}}_\mu^{\mathrm{T}}\Big(\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_\mu^{\mathrm{T}}\Big)^{-1}\dot{\hat{\mathbf{G}}}\dot{\mathbf{X}}. \tag{3.32}$$

*Under the same additional assumption* (2.9) of the previous procedure, the projection of (3.31) on the directions tangential to the constraints and the consideration of the generalized independent velocities $\dot{\mathbf{X}}^*$ results in

$$\mathbf{M}^{\#}\ddot{\mathbf{X}}^* + \mathbf{B}^{\#} = \mathbf{0} \tag{3.33}$$

with

$$\mathbf{M}^{\#} = \mathbf{M}^{\#}(\mathbf{X}) = \mathbf{CMC}^{\mathrm{T}} \tag{3.34}$$

and

$$\mathbf{B}^{\#} = \mathbf{B}^{\#}\left(\mathbf{X}, \dot{\mathbf{X}}^{*}\right) = \mathbf{C}\left(\bar{\mathbf{B}}\left(\mathbf{X}, \mathbf{C}^{\mathrm{T}}\dot{\mathbf{X}}^{*}\right) + \mathbf{M}\dot{\mathbf{C}}^{\mathrm{T}}\dot{\mathbf{X}}^{*}\right). \tag{3.35}$$

Notice that the latter procedure for the elimination of the reactions yields an effective mass matrix $\mathbf{M}^{\#}$ (3.33) that is not affected by the friction coefficient and an effective vector of nonlinear terms that combines the original vector $\mathbf{B}$ with mass and friction effects.

Existence and uniqueness of solution to the problems (3.25), (3.27), or (3.25), (3.33), both with initial conditions (3.1), (3.2), follows from the general theory of ordinary differential equations, provided the matrices $\mathbf{M}^{*}$ or $\mathbf{M}^{\#}$ are invertible and the functions $\mathbf{M}(\mathbf{X}), \mathbf{B}(\mathbf{X}, \dot{\mathbf{X}})$ and $\mathbf{G}(\mathbf{X})$ are sufficiently regular. In addition, the accelerations $\ddot{\mathbf{X}}(t)$ and the reactions $\hat{\mathbf{r}}(t)$ depend continuously on time. Then, in view of the strict inequalities in (3.15), (3.16) and (3.17), no change in the state of any contact candidate particle occurs in some interval $]\tau, \tau + \Delta\tau[$, which justifies the assumptions made above and confirms the solution to the problem (3.19) and (3.20) as a solution to the original problem (2.13), (2.15) and (2.16).

In this manner we have proved the following result:

**Proposition 3.6.** Assume that:

(i) the lines of $\mathbf{G}(\mathbf{X})$ are linearly independent (2.9);
(ii) the equivalent conditions (3.5) and (3.11) hold with $\mathbf{G}^{\mathrm{T}}\Delta\mathbf{r} = \mathbf{MA} \neq \mathbf{0}$;
(iii) for each particle in $P_{\mathrm{c}}$, one of the three sets of conditions (3.15), (3.16) or (3.17) holds;
(iv) matrices $\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_{\mu}^{\mathrm{T}}$ in (3.30) and $\mathbf{M}^{*}$ in (3.28) are invertible (see Lemma 3.7 below);
(v) $\mathbf{M}(\mathbf{X})$, $\mathbf{B}(\mathbf{X}, \dot{\mathbf{X}})$, $\mathbf{G}(\mathbf{X})$ and $\partial\mathbf{G}(\mathbf{X})/\partial\mathbf{X}$ are bounded and Lipschitz continuous in $(\mathbf{X}, \dot{\mathbf{X}})$.

Then there exists a dynamic solution with an initial acceleration and reaction discontinuity followed by a smooth dynamic solution. □

In relation with the above procedures for elimination of the reactions $\hat{\mathbf{r}}$ we observe the following.

**Lemma 3.7.** Let (2.9) hold. Then the matrix $\mathbf{M}^{*}$ in (3.28) is singular if and only if the matrix $\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_{\mu}^{\mathrm{T}}$ in (3.30) is singular.

**Proof.** First we prove that if $\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_{\mu}^{\mathrm{T}}$ is singular then $\mathbf{M}^{*}$ is singular. Let $\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_{\mu}^{\mathrm{T}}\hat{\mathbf{r}}' = \hat{\mathbf{G}}\mathbf{M}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{H}_{\mu}\hat{\mathbf{r}}' = \mathbf{0}$ for some $\mathbf{0} \neq \hat{\mathbf{r}}' \in \mathbb{R}^{\hat{n}}$. Since $\mathbf{H}_{\mu}$ and $\mathbf{G}^{\mathrm{T}}$ are full column rank matrices, then $\mathbf{0} \neq \mathbf{M}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{H}_{\mu}\hat{\mathbf{r}}' \in Ker(\hat{\mathbf{G}})$. For $\mathbf{M}^{*}$ to be singular

$$\exists \mathbf{A}' \in Ker\left(\hat{\mathbf{G}}\right), \quad \mathbf{A}' \neq \mathbf{0}: \quad \left(\mathbf{I} - \mathbf{G}^{\mathrm{T}}\mathbf{H}_{\mu}\hat{\mathbf{I}}\mathbf{G}^{\dagger}\right)\mathbf{MA}' \cdot \mathbf{A}'' = 0, \quad \forall \mathbf{A}'' \in Ker\left(\hat{\mathbf{G}}\right). \tag{3.36}$$

The latter condition is fulfilled with $\mathbf{A}'$ defined as $\mathbf{M}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{H}_{\mu}\hat{\mathbf{r}}'$, because $\mathbf{G}^{\dagger}\mathbf{G}^{\mathrm{T}} = \mathbf{I}$ and $\mathbf{H}_{\mu}\hat{\mathbf{I}}H_{\mu} = \mathbf{H}_{\mu}$.

We prove now that if $\mathbf{M}^{*}$ is singular then $\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_{\mu}^{\mathrm{T}}$ is singular. We assume that (3.36) holds. Consequently

$$\mathbf{MA}' - \mathbf{G}^{\mathrm{T}}\mathbf{H}_{\mu}\hat{\mathbf{I}}\mathbf{G}^{\dagger}\mathbf{MA}' \in \left[Ker\left(\hat{\mathbf{G}}\right)\right]^{\perp} = Rg\left(\hat{\mathbf{G}}^{\mathrm{T}}\right) \subset Rg\left(\mathbf{G}^{\mathrm{T}}\right) \tag{3.37}$$

and also $\mathbf{0} \neq \mathbf{MA}' \in Rg(\mathbf{G}^{\mathrm{T}})$. The $\mathbf{0} \neq \mathbf{MA}' = \mathbf{G}^{\mathrm{T}}\mathbf{r}'$ with $\mathbf{0} \neq \mathbf{r}' \in \mathbb{R}^{2n_{\mathrm{C}}}$ due to (2.9). From (3.37) we conclude that

$$\mathbf{G}^{\mathrm{T}}\begin{bmatrix} \mathbf{r}'_{free} \\ \mathbf{r}'_{\mathrm{t}\,slip} - \mu\mathbf{S}_{slip}\mathbf{r}'_{\mathrm{n}\,slip} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \mathbf{G}^{\mathrm{T}}\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{y}_{\mathrm{n}\,slip} \\ \mathbf{y}_{stick} \end{bmatrix}$$

for some $\mathbf{y}_{\mathrm{n}\,slip}$ and $\mathbf{y}_{stick}$. Due to (2.9), the previous equation implies that $\mathbf{r}'_{free}$, $\mathbf{r}'_{\mathrm{t}\,slip} - \mu\mathbf{S}_{slip}\mathbf{r}'_{\mathrm{n}\,slip}$, $\mathbf{y}_{\mathrm{n}\,slip}$ and $\mathbf{y}_{stick}$ vanish. Hence $\mathbf{0} \neq \mathbf{r}' = \mathbf{H}_{\mu}\hat{\mathbf{I}}\mathbf{r}' = \mathbf{H}_{\mu}\hat{\mathbf{r}}'$, which implies $\hat{\mathbf{r}}' \neq \mathbf{0}$, since $\mathbf{H}_{\mu}$ is a full column rank matrix. Finally we conclude that

$$\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_{\mu}^{\mathrm{T}}\hat{\mathbf{r}}' = \hat{\mathbf{G}}\mathbf{M}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{H}_{\mu}\hat{\mathbf{r}}' = \mathbf{0},$$

because $\mathbf{M}^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{H}_{\mu}\hat{\mathbf{r}}' = \mathbf{A}' \in Ker\,(\hat{\mathbf{G}})$. □

### *3.2. A three degree-of-freedom example of initial acceleration and reaction discontinuities*

We consider a homogeneous thin rigid rod of length $L$ and mass $m$ with plane motion, that may establish frictional contact with a fixed flat obstacle at the extremity A of the rod (see Fig. 1). In absence of contact the system has three degrees of freedom; the adopted generalized coordinates are the angle $\theta$ between the rod and the vertical direction and the $X$ and $Y$ coordinates of the center of mass $CM$, which are grouped in the vector $\mathbf{X} = [\theta \quad X \quad Y]^{\mathrm{T}}$. Constant external forces, $F_X$ and $F_Y$, and moment, $M_z$, are applied at the rod center of mass. The kinematically admissible half-plane is defined by $\phi(\mathbf{X}) = -Y + L/2\cos\theta \leqslant 0$.

Numerous studies have considered this or related systems. The interest has been concentrated on: (i) the non-existence or the non-uniqueness of solution to the problem of finding the (right) accelerations and reactions of the rod for some initial conditions involving non-vanishing velocities (Lötstedt, 1981; Pfeiffer and Glocker, 1996; Génot and Brogliato, 1998); (ii) the occurrence of velocity discontinuities

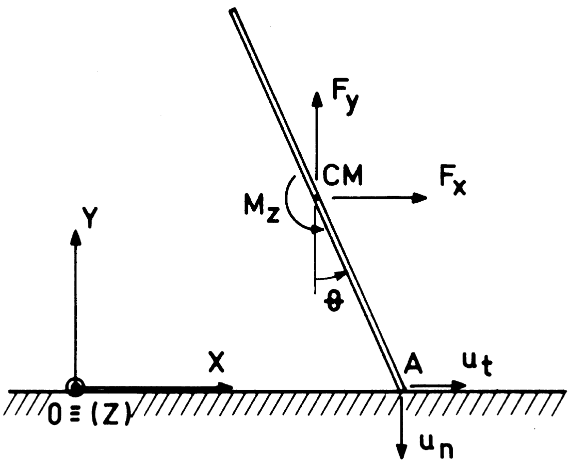


Fig. 1. A thin homogeneous rod subjected to constant forces at its center of mass (CM) and frictional contact reactions at its lowest extremity (A).

during an impact (Jean and Moreau, 1985; Brach, 1989; Stronge, 1990; Wang and Mason, 1992; Stewart and Trinkle, 1996); (iii) the possible occurrence of velocity discontinuities without any impact (frictional catastrophes, Moreau, 1988) for some initial conditions that also involve a non-vanishing kinetic energy (Mason and Wang, 1998). Our interest here is to study the occurrence of dynamic solutions initiating at some equilibrium states of the rod with an acceleration and reaction discontinuity; vanishing initial velocities are considered in the present study, a situation addressed also in Example 2 of Lötstedt (1981).

We consider equilibrium states such that $\phi(\mathbf{X}^0) = -Y^0 + L/2\cos\theta^0 = 0$ and $\mathbf{r}^0 = [r_t^0 \quad r_n^0]^T = [\alpha - 1]^T$, with $-\mu \leqslant \alpha \leqslant 0$. We study separately the case of a static reaction in the interior of the friction cone (Case 1, $-\mu < \alpha \leqslant 0$) and the case of a non-vanishing reaction on the friction cone (Case 2, $\alpha = -\mu$). The analysis for $0 \leqslant \alpha \leqslant \mu$ would be similar, and it is easy to show that no initial acceleration and reaction discontinuities are possible when $\mathbf{r}^0 = \mathbf{0}$ and $\phi(\mathbf{X}^0) \leqslant 0$. The right generalized accelerations are denoted by $\ddot{\theta}^+$, $\ddot{X}^+$ and $\ddot{Y}^+$ and the right normal and tangential acceleration of particle A are denoted by $a_n^+$ and $a_t^+$. The jump eqns (3.5) are

$$\begin{bmatrix} \frac{mL^2}{12} & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\theta}^+ \\ \ddot{X}^+ \\ \ddot{Y}^+ \end{bmatrix} = \mathbf{G}^T(\mathbf{X}^0) \begin{bmatrix} r_t^+ - \alpha \\ r_n^+ + 1 \end{bmatrix}$$

where

$$\mathbf{G}(\mathbf{X}) = \begin{bmatrix} \mathbf{G}_t^A \\ \mathbf{G}_n^A \end{bmatrix} = \begin{bmatrix} \frac{L}{2}\cos\theta & 1 & 0 \\ -\frac{L}{2}\sin\theta & 0 & -1 \end{bmatrix}.$$

**Case 1** [*static reaction in the interior of the friction cone:* $-\mu < \alpha \leqslant 0$]. In this case, it is possible to show that a reaction jump towards a contact state with non-vanishing reaction and possible slip to the

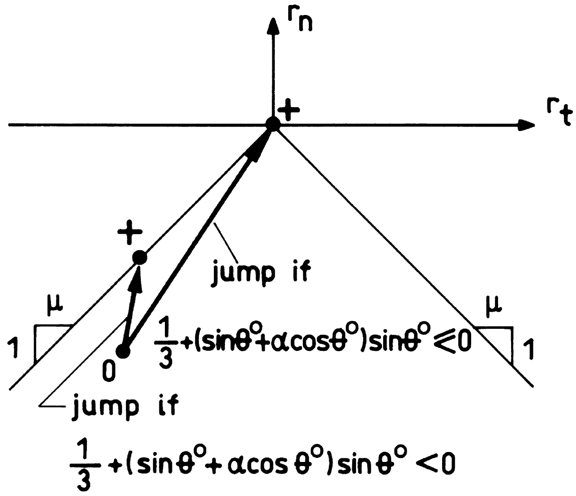


Fig. 2. Possible initial reaction jumps followed by smooth near future evolutions in Case 1 (static reaction in the interior of the friction cone).

left ($r_n^+ < 0$ and $r_t^+ = -\mu r_n^+$) cannot occur. The possible cases of initial acceleration and reaction discontinuities are discussed next (see Fig. 2).

**Case 1A** [*jumps towards a contact state with non-vanishing reaction and possible slip to the right:* $a_t^+ > 0, a_n^+ = 0, r_n^+ < 0, r_t^+ = \mu r_n^+$]. From the jump equations we obtain

$$a_t^+ = -\frac{\frac{4}{3}(\mu + \alpha)}{m\left[\frac{1}{3} + \left(\sin\theta^0 - \mu\cos\theta^0\right)\sin\theta^0\right]}$$

and

$$r_n^+ = -\frac{\frac{1}{3} + \left(\sin\theta^0 + \alpha\cos\theta^0\right)\sin\theta^0}{\frac{1}{3} + \left(\sin\theta^0 - \mu\cos\theta^0\right)\sin\theta^0}$$

so that a necessary condition for such a jump is that $1/3 + (\sin\theta^0 + \alpha\cos\theta^0)\sin\theta^0 < 0$, which occurs only if $\alpha < 0$. Notice that this jump occurs with a reduction of the absolute values of both the tangential and the normal reaction.

The consideration of the equations of motion for a smooth near future sliding to the right yields, after eliminating the translational degrees-of-freedom,

$$\frac{mL}{2}\left[\frac{1}{3} + (\sin\theta - \mu\cos\theta)\sin\theta\right]\ddot{\theta}$$
$$= -\frac{mL}{2}\dot{\theta}^2(\sin\theta - \mu\cos\theta)\cos\theta - \sin\theta^0 - \alpha\cos\theta^0 + \sin\theta - \mu\cos\theta.$$

Since $1/3 + (\sin\theta^0 - \mu\cos\theta^0)\sin\theta^0 < 0$, particle A does begin sliding towards the right. Notice that, in this example, the matrix $\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_\mu^T$ corresponding to strict sliding towards the right reduces to a scalar that is equal to the product of $1/3 + (\sin\theta - \mu\cos\theta)\sin\theta$ by a positive constant factor. This means that all the assumptions of Proposition 3.6 hold in this case, so that the previous conclusion that a smooth evolution exists after the initial acceleration and reaction jump is also the result of applying that proposition.

We determine now the motion of the rod for a specific set of data suitable for the present case: $L = 1$, $m = 1$, $\mu = 2$, $\mathbf{X}^0 = [0.5 \quad 0 \quad 0.439]^T$ and $\mathbf{r}^0 = [-1.8 \quad -1]^T$. Immediately after the jump the reactions are $\mathbf{r}^+ = [-1.395 \quad -0.698]^T$ and the extremity A begins sliding towards the right with increasing contact reactions. After some time, its horizontal velocity vanishes. At this instant, $\mathbf{X} = [0.645 \quad 0.03 \quad 0.4]^T$, $\dot{\mathbf{X}} = [0.139 \quad -0.056 \quad -0.042]^T$ and $\mathbf{r} = [-2.458 \quad -1.229]^T$. The analysis of the equations of motion for the possible near future evolutions leads to the conclusion that contact is lost: a reaction jump towards the vertex of the friction cone occurs when the horizontal velocity of particle A vanishes. A stroboscopic representation of the motion is shown in Fig. 3.

**Case 1B** [*jump towards a contact state with no reaction:* $a_n^+ \leqslant 0, \mathbf{r}^+ = \mathbf{0}$]. The expression for the normal acceleration of the contact extremity of the rod is

$$a_n^+ = \frac{1}{m}\left[1 + 3\left(\sin\theta^0 + \alpha\cos\theta^0\right)\sin\theta^0\right].$$

A necessary condition for such a jump is that $1/3 + (\sin\theta^0 + \alpha\cos\theta^0)\sin\theta^0 \leqslant 0$. From the equations of motion for a smooth evolution in the near future, we conclude that two different possibilities exist:

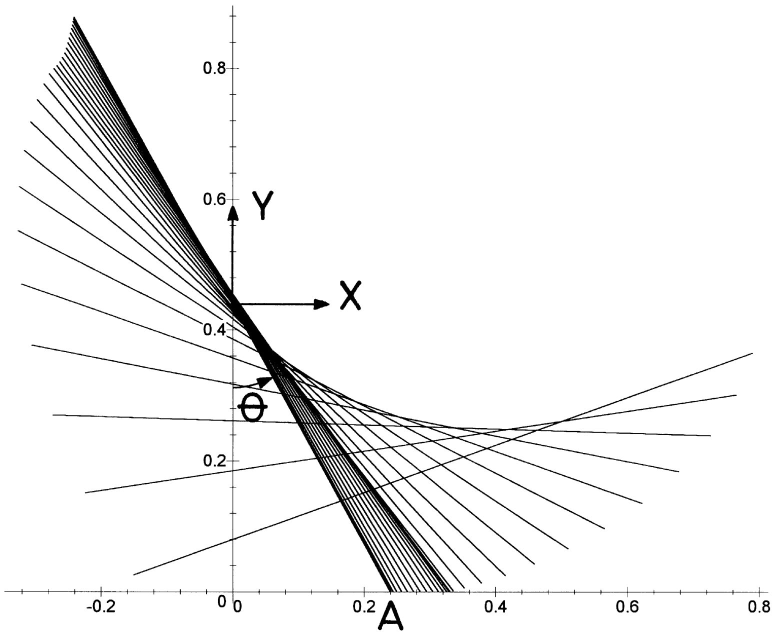


Fig. 3. Stroboscopic view of the motion of the rod after an initial acceleration and reaction discontinuity towards a contact state with non-vanishing reaction and possible slip to the right. Data: $L = 1$, $m = 1$, $\mu = 2$, $\mathbf{X}^0 = [0.5\ 0\ 0.439]^{\mathrm{T}}$ and $\mathbf{r}^0 = [-1.8\ -1]^{\mathrm{T}}$.

(i) if $1/3 + (\sin\theta^0 + \alpha\cos\theta^0)\sin\theta^0 < 0$, the right normal acceleration the particle A is strictly negative ($a_{\mathrm{n}}^+ < 0$) and the rod looses contact, a conclusion that is consistent with the result of applying Proposition 3.6 to this case; in Fig. 4 a stroboscopic representation of the motion of the rod for a specific set of data is presented;

(ii) if $1/3 + (\sin\theta^0 + \alpha\cos\theta^0)\sin\theta^0 = 0$, the right normal acceleration of particle A vanishes but the lowest order derivative of the displacement of particle A along the normal to the obstacle that does not vanish (the fourth derivative) is strictly negative if $\sin 2\theta^0 + \alpha\cos 2\theta^0 < 0$ (the rod loses contact), and strictly positive if $\sin 2\theta^0 + \alpha\cos 2\theta^0 > 0$ (contact persists but with sliding towards the right); note that Proposition 3.6 could not be applied to this case because assumption (iii) does not hold.

**Case 2** [*non-vanishing reaction on the friction cone*: $\alpha = -\mu$]. Similarly to Case 1, it can be shown that no jump is possible towards a contact state with non-vanishing reaction and possible slip to the left $r_{\mathrm{n}}^+ < 0$ and $r_{\mathrm{t}}^+ = -\mu r_{\mathrm{n}}^+$). It can also be shown that no jump is possible towards a contact state with non-vanishing reaction and possible slip to the right. The necessary condition for the occurrence of such a jump is that $1/3 + (\sin\theta^0 - \mu\cos\theta^0)\sin\theta^0 = 0$. But the solution of the equations of motion for a smooth near future sliding towards the right yields indeterminacies of the type 0/0 for the tangential acceleration of the contact particle A and for the normal reaction, as the initial time (the instant of the possible discontinuity) is approached from the right. The evaluation of those limits with l'Hôpital's rule yields always a sign inconsistency for the normal reaction or for the tangential acceleration of particle A. This example shows that the verification of the necessary condition for an acceleration jump does not guarantee that it can actually be followed by a smooth non-trivial dynamic solution. Notice that assumption (iv) of Proposition 3.6 does not hold in this case. The only possible case of initial acceleration and reaction discontinuities is the following (see Fig. 5).

**Case 2A** [*jump towards a no reaction contact state*: $a_{\mathrm{n}}^+ \leqslant 0$ and $\mathbf{r}^+ = \mathbf{0}$]. The expressions for the normal and tangential acceleration of the contact particle A are:

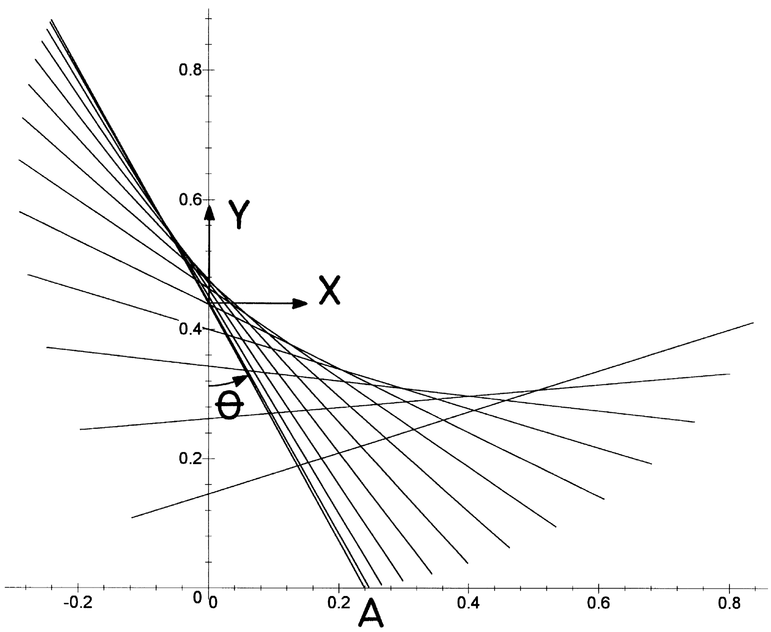


Fig. 4. Stroboscopic view of the motion of the rod after an initial acceleration and reaction discontinuity towards a contact state with no reaction, followed by loss of contact. Data: $L = 1$, $m = 1$, $\mu = 2$, $\mathbf{X}^0 = [0\ 0.439\ 0.5]^{\mathrm{T}}$ and $\mathbf{r}^0 = [-1.8\ -1]^{\mathrm{T}}$.

$$a_{\mathrm{n}}^{+} = \frac{1}{m}\left[1 + 3\left(\sin\theta^0 - \mu\cos\theta^0\right)\sin\theta^0\right];$$

$$a_{\mathrm{t}}^{+} = -\frac{1}{m}\left[-\mu + 3\left(\sin\theta^0 - \mu\cos\theta^0\right)\cos\theta^0\right].$$

A necessary condition for such a jump is that $1/3 + (\sin\theta^0 - \mu\cos\theta^0)\sin\theta^0 \leqslant 0$. Two possibilities exist:

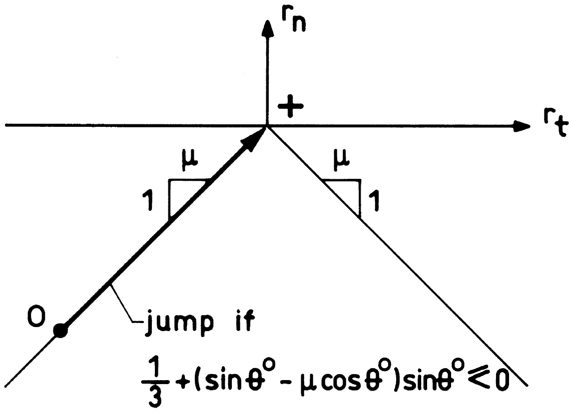


Fig. 5. Possible initial reaction jumps followed by smooth near future evolutions in Case 2 (non-vanishing static reaction on the friction cone).

(i) if $1/3+(\sin\theta^0-\mu\cos\theta^0)\sin\theta^0 < 0$, the normal acceleration of particle A is strictly negative and the rod looses contact: Proposition 3.6 could be applied to this case; (ii) if $1/3+(\sin\theta^0-\mu\cos\theta^0)\sin\theta^0=0$, the right normal acceleration of particle A vanishes and assumption (iii) of Proposition 3.6 does not hold; the lowest order derivative of the displacement of particle A along the normal to the obstacle that does not vanish (the fourth derivative) is strictly negative if $\sin 2\theta^0-\mu\cos 2\theta^0 < 0$ (the rod looses contact), and strictly positive if $\sin 2\theta^0-\mu\cos 2\theta^0>0$; in the latter case the above right acceleration $a_t^+$ would be positive but the required normal reaction would be also positive, so that a smooth near future sliding towards the right is not possible (assumption (iv) of Proposition 3.6 does not hold).

## 4. Divergence of smooth dynamic solutions with perturbed initial conditions

We discuss now the existence of smooth dynamic solutions $(\mathbf{X}(t), \mathbf{r}(t))$ to the nonlinear system (2.13) starting from (perturbed) initial conditions arbitrarily close to an equilibrium state $(\mathbf{X}^0, \mathbf{r}^0)$ and diverging from that equilibrium state. Liapunov's linearization method is used for that purpose: the stability of the actual nonlinear system is studied by analysing the behavior of the *directionally* linearized system in the neighborhood of the static equilibrium configuration. In particular, we will show in Proposition 4.4 that, under certain conditions, a solution to the nonlinear system for an initial perturbation along a direction that corresponds to a divergent solution of the linearized system, is also divergent.

Throughout this section we assume that

$$\mathbf{M}(\mathbf{X}), \mathbf{B}(\mathbf{X}, \dot{\mathbf{X}}), \mathbf{G}(\mathbf{X}) \textit{ and } \partial\mathbf{G}(\mathbf{X})/\partial\mathbf{X} \textit{ are continuously differentiable in the neighborhood of } (\mathbf{X}, \dot{\mathbf{X}}) = (\mathbf{X}^0, \mathbf{0}). \tag{4.1}$$

### 4.1. The directionally linearized problem

In the neighborhood of the equilibrium state, and for admissible directions of the increments of the generalized coordinates and the contact reactions

$$\delta\mathbf{X}(t)\in K_{\mathrm{V}}(\mathbf{X}^0, \mathbf{r}^0), \quad \delta\mathbf{r}(t)\in K_{\mathbf{w}}(\mathbf{X}^0, \mathbf{r}^0, \delta\mathbf{X}(t)), \tag{4.2}$$

the equations of motion (2.13) have the following linearized form

$$\mathbf{M}(\mathbf{X}^0)\delta\ddot{\mathbf{X}}(t)+\mathbf{K}(\mathbf{X}^0, \mathbf{r}^0)\delta\mathbf{X}(t)=\mathbf{G}^{\mathrm{T}}(\mathbf{X}^0)\delta\mathbf{r}(t), \tag{4.3}$$

where the $N\times N$ *tangent stiffness matrix* $\mathbf{K}(\mathbf{X}^0, \mathbf{r}^0)$ is the sum of contributions from the deformation energy, the potential energy of the external forces and the contact related nonlinearities, respectively:

$$\mathbf{K}(\mathbf{X}^0, \mathbf{r}^0)=\mathbf{K}^{\mathrm{U}}(\mathbf{X}^0)+\mathbf{K}^{\Omega}(\mathbf{X}^0)+\mathbf{K}^{\mathrm{C}}(\mathbf{X}^0, \mathbf{r}^0), \tag{4.4}$$

where, for $i, j=1, \ldots, N$,

$$K_{ij}^{\mathrm{U}}(\mathbf{X}^0)=-\frac{\partial F_i^{\mathrm{U}}}{\partial X_j}(\mathbf{X}^0)=\frac{\partial^2\mathrm{U}}{\partial X_i\partial X_j}(\mathbf{X}^0), \quad K_{ij}^{\Omega}(\mathbf{X}^0)=-\frac{\partial F_i^{\Omega}}{\partial X_j}(\mathbf{X}^0)=\frac{\partial^2\mathbf{\Omega}}{\partial X_i\partial X_j}(\mathbf{X}^0)$$

$$K_{ij}^{\mathrm{C}}(\mathbf{X}^0, \mathbf{r}^0)=-\sum_{P_c}\left[\left(r_{\mathrm{n}}^0\mathbf{n}+r_{\mathrm{t}}^0\mathbf{t}\right)\cdot\frac{\partial^2\mathbf{x}}{\partial X_i\partial X_j}-\chi\left(r_{\mathrm{n}}^0 G_{\mathrm{t}i}G_{\mathrm{t}j}-r_{\mathrm{t}}^0 G_{\mathrm{n}i}G_{\mathrm{t}j}\right)\right]. \tag{4.5}$$

The latter contribution to the tangent stiffness matrix and the right-hand side of (4.3) result both from the linearization of the generalized reactions $\mathbf{R}(\mathbf{X}, \mathbf{r}) = \mathbf{G}^{\mathrm{T}}(\mathbf{X})\mathbf{r}$:

$$\delta\mathbf{R}(\mathbf{X}^0, \mathbf{r}^0) = \delta\mathbf{G}^{\mathrm{T}}(\mathbf{X}^0)\mathbf{r}^0 + \mathbf{G}^{\mathrm{T}}(\mathbf{X}^0)\delta\mathbf{r}$$

$$= -\mathbf{K}^{\mathrm{C}}(\mathbf{X}^0, \mathbf{r}^0)\delta\mathbf{X} + \mathbf{G}^{\mathrm{T}}(\mathbf{X}^0)\delta\mathbf{r}. \tag{4.6}$$

From (4.5) it is clear that the first contribution to the matrix $\mathbf{K}^{\mathrm{C}}$ has the character of a geometric stiffness matrix and the second contribution results from the obstacle curvature. This latter contribution may be nonsymmetric only if tangential reactions exist on the contact. Note that the nonlinear inertia terms $\mathbf{D}$ in (2.13) and (2.14) have no contribution to the linearized equations at $(\mathbf{X}, \dot{\mathbf{X}}) = (\mathbf{X}^0, \mathbf{0})$, since they are quadratic functions of the generalized velocities.

### 4.2. Instability of the directionally linearized dynamic problem

For $t \geqslant \tau$ we consider perturbed dynamic solutions of the linearized system (4.3) in the form

$$\delta\mathbf{X}(t) = \alpha(t)\mathbf{V}, \quad \delta\mathbf{r}(t) = \beta(t)\mathbf{w}, \tag{4.7}$$

where

$$\mathbf{V} \in K_{\mathrm{V}}(\mathbf{X}^0, \mathbf{r}^0) \quad \text{and} \quad \mathbf{w} \in K_{\mathbf{w}}(\mathbf{X}^0, \mathbf{r}^0, \mathbf{V}) \tag{4.8}$$

define constant directions in the sets of right admissible displacement and reaction increments; the function of time $\alpha$ is twice continuously differentiable and $\alpha$ and $\dot{\alpha}$ are non-negative and non-decreasing; the function $\beta$ is continuous, non-negative and non-decreasing in the same interval; the initial values $\alpha(\tau) \geqslant 0$ and $\dot{\alpha}(\tau) \geqslant 0$ are arbitrarily small.

For each $(\mathbf{V}, \mathbf{V}') \in \mathbb{R}^N \times \mathbb{R}^N$ we continue to use the notation (3.3) and we define:

$$a^*(\mathbf{V}, \mathbf{V}') \stackrel{\text{def}}{=} \mathbf{KV} \cdot \mathbf{V}' - \sum_{P_{\mathrm{c}}} \mu\gamma_{\mathrm{n}}^{\dagger}(\mathbf{KV})|\gamma_{\mathrm{t}}(\mathbf{V}')|. \tag{4.9}$$

For given $\mathbf{V} \in \mathbb{R}^N$ and each $(\mathbf{w}, \mathbf{w}') \in \mathbb{R}^{2n_{\mathrm{C}}} \times \mathbb{R}^{2n_{\mathrm{C}}}$ we also define

$$m_{\mathbf{V}}^{\#}(\mathbf{w}, \mathbf{w}') \stackrel{\text{def}}{=} \mathbf{GM}^{-1}(\mathbf{G}^{\mathrm{T}}\mathbf{w} - \mathbf{KV}) \cdot \mathbf{w}' - \sum_{P_{\mathrm{c}}} \mu\left|\gamma_{\mathrm{t}}\left(\mathbf{M}^{-1}(\mathbf{G}^{\mathrm{T}}\mathbf{w} - \mathbf{KV})\right)\right| w_{\mathrm{n}}'. \tag{4.10}$$

Note that, for $\mathbf{V} = \mathbf{0}$, the latter definition reduces to the definition (3.4) of $m^{\#}(\mathbf{w}, \mathbf{w}')$.

**Proposition 4.1.** Let (2.9) hold. Then

$\exists\lambda \geqslant 0$ and $\mathbf{V} \in K_{\mathrm{V}}(\mathbf{X}^0, \mathbf{r}^0)$, $\mathbf{V} \neq \mathbf{0}$, such that

$$(\lambda^2\mathbf{M} + \mathbf{K})\mathbf{V} \in \mathbf{G}^{\mathrm{T}}K_{\mathbf{w}}(\mathbf{X}^0, \mathbf{r}^0, \mathbf{V}) \tag{4.11}$$

i.e., such that

$$\lambda^2\left[m^*(\mathbf{V}, \mathbf{V}') - m^*(\mathbf{V}, \mathbf{V})\right] + \left[a^*(\mathbf{V}, \mathbf{V}') - a^*(\mathbf{V}, \mathbf{V})\right] \geqslant 0, \quad \forall\mathbf{V}' \in K_{\mathrm{V}}(\mathbf{X}^0, \mathbf{r}^0), \tag{4.12}$$

if and only if the linearized system (4.3) admits a solution of the form (4.7) and (4.8) with

$$\mathbf{w} = \mathbf{G}^{\dagger}\left(\lambda^2 \mathbf{M} + \mathbf{K}\right)\mathbf{V} \tag{4.13}$$

and

$$\beta(t) = \alpha(t) = \begin{cases} \alpha(\tau)\cosh\left[\lambda(t-\tau)\right] + \left[\dfrac{\dot{\alpha}(\tau)}{\lambda}\right]\sinh\left[\lambda(t-\tau)\right] & \text{if} \quad \lambda > 0 \\ \alpha(\tau) + \dot{\alpha}(\tau)(t-\tau) \quad \text{if} \ \lambda = 0. \end{cases} \tag{4.14}$$

In these circumstances, the equilibrium state ($\mathbf{X}^0$, $\mathbf{r}^0$) of the linearized system is dynamically unstable by divergence.

**Proof.** The equivalence between (4.11) and (4.12) results from the characterization (2.37) of the set $\mathbf{G}^{\mathrm{T}} K_{\mathbf{w}}(\mathbf{X}^0, \mathbf{r}^0, \mathbf{V})$. The sufficiency of (4.11) is obtained by direct substitution of (4.7), (4.13) and (4.14) in (4.3). The necessity part of the proof follows the same arguments of Propositions 3.7 and 3.9 in Martins et al. (1998). □

An alternative formulation for the above eigenproblem can be given. With that purpose, we consider first the following auxiliary problem.

Given $\mathbf{V} \in \mathbb{R}^N$, find $\mathbf{w}_{\mathbf{V}} \in K_{\mathbf{w}}(\mathbf{X}^0, \mathbf{r}^0)$ such that

$$\mathbf{M}^{-1}\left(\mathbf{G}^{\mathrm{T}}\mathbf{w}_{\mathbf{V}} - \mathbf{K}\mathbf{V}\right) \in K_{\mathrm{V}}\left(\mathbf{X}^0, \mathbf{r}^0, \mathbf{w}\right), \tag{4.15}$$

i.e., such that

$$m_{\mathbf{v}}^{\#}\left(\mathbf{w}_{\mathbf{V}}, \mathbf{w}' - \mathbf{w}_{\mathbf{V}}\right) \geqslant 0, \quad \forall \mathbf{w}' \in K_{\mathbf{w}}\left(\mathbf{X}^0, \mathbf{r}^0\right). \tag{4.16}$$

If (2.9) holds and the coefficient of friction is sufficiently small, it is possible to show that, for each given $\mathbf{V} \in \mathbb{R}^N$, there is a unique solution $\mathbf{w}_{\mathbf{V}}$ to the problem (4.15) and (4.16); the proof can be done by using arguments of the type used by Klarbring (1990b) in a related problem; see also Cocu (1990). In these circumstances, we denote by $\mathscr{W}$: $\mathbb{R}^N$: $\to K_{\mathbf{w}}(\mathbf{X}^0, \mathbf{r}^0)$ the map that assigns to each $\mathbf{V} \in \mathbb{R}^N$ the unique solution $\mathbf{w}_{\mathbf{V}}$ to the problem (4.15) and (4.16). Then we construct the *nonlinear equality* eigenproblem

$$\lambda^2 \mathbf{M}\mathbf{V} + \mathbf{K}\mathbf{V} - \mathbf{G}^{\mathrm{T}}\mathscr{W}(\mathbf{V}) = \mathbf{0}, \tag{4.17}$$

and we have the following result.

**Proposition 4.2.** Assume that (2.9) holds and that the problem (4.15) and (4.16) has a unique solution. If there exists $\mathbf{V} \in \mathbb{R}^N$, $\mathbf{V} \neq \mathbf{0}$, such that (4.17) holds, with $\lambda > 0$, then the linearized system (4.3) admits a solution of the form (4.7) and (4.8) with $\mathbf{w} = \mathscr{W}(\mathbf{V})$ and $\alpha$ and $\beta$ given by $(4.14)_1$. The same conclusion holds, with $\alpha$ and $\beta$ given by $(4.14)_2$, if (4.17) is satisfied with $\lambda = 0$ and $\mathbf{0} \neq \mathbf{V} \in K_{\mathrm{V}}(\mathbf{X}^0, \mathbf{r}^0, \mathbf{w})$.

**Proof.** From (4.17) and (4.15), it is clear that

$$\mathscr{W}(\mathbf{V}) \in K_{\mathbf{w}}\left(\mathbf{X}^0, \mathbf{r}^0\right)$$

$$\lambda^2 \mathbf{V} = \mathbf{M}^{-1}\left(\mathbf{G}^{\mathrm{T}}\mathscr{W}(\mathbf{V}) - \mathbf{K}\mathbf{V}\right) \in K_{\mathrm{V}}\left(\mathbf{X}^0, \mathbf{r}^0, \mathscr{W}(\mathbf{V})\right), \tag{4.18}$$

from which it follows, if $\lambda > 0$, that $\mathbf{V} \in K_{\mathrm{V}}(\mathbf{X}^0, \mathbf{r}^0, \mathscr{W}(\mathbf{V}))$. Hence (4.11) holds.

If $\lambda = 0$, $\mathbf{V} \in K_V(\mathbf{X}^0, \mathbf{r}^0, \mathscr{W}(\mathbf{V}))$ has to be imposed independently since it does not necessarily follow from (4.18). □

The above conditions for divergence instability lead to the resolution of *a set* of classical generalized *linear* eigenproblems, together with the verification of some inequalities. These various linear eigenproblems are obtained by considering all possible combinations of near future states of the contact candidate particles; actually, since the particles in $P_f(\mathbf{X}^0)$ (currently free) and in $P_d(\mathbf{X}^0, \mathbf{r}^0)$ (currently stick) will remain so in the near future, only the combinations of possible near future states of the particles in $P_z(\mathbf{X}^0, \mathbf{r}^0)$ and $P_s(\mathbf{X}^0, \mathbf{r}^0)$ need to be considered.

The possible near future evolutions for the particles in $P_z(\mathbf{X}^0, \mathbf{r}^0)$ and $P_s(\mathbf{X}^0, \mathbf{r}^0)$ are summarized in Table 2.

The total number of classical generalized linear eigenproblems that may be constructed in this manner is $4^{n_z} \times 2^{n_s}$ ($n_z = \#P_z$, $n_s = \#P_s$), because a contact particle in $P_z$ has four possible near future evolutions (zf, zd, zs+ and zs−) and a particle in $P_s$ has two possible near future evolutions (sd and ss). Assuming now that *one* combination of the above near future evolutions holds, and taking into account all the *equality* conditions in columns I and II of Table 2, the inclusion eigenproblem (4.11) becomes the following constrained linear eigenproblem on the unknowns $(\lambda^2, \mathbf{V}, \hat{\mathbf{w}})$

$$\left(\lambda^2\mathbf{M} + \mathbf{K}\right)\mathbf{V} = \hat{\mathbf{G}}_\mu^{\mathrm{T}}\hat{\mathbf{w}} \tag{4.19}$$

$$\hat{\mathbf{G}}\mathbf{V} = \mathbf{0}, \tag{4.20}$$

followed by the verification that the solution vectors $(\mathbf{V}, \hat{\mathbf{w}})$ satisfy the inequalities in columns III and IV of Table 2. The decompositions used in (3.18) are used in (4.19) and (4.20) for the vector $\mathbf{w}$ and the matrix $\mathbf{G}$. Notice that here we have

Table 2
Conditions on $\mathbf{V}$ and $\mathbf{w}$ corresponding to every possible near future evolution of a contact particle that belongs to set $P_z$ or $P_s$ in the static equilibrium state

| | | Equality conditions imposed on the construction of the linear eigenproblem | | Remaining inequality conditions to be verified by the solution | |
|---|---|---|---|---|---|
| | | I kinematic | II static | III kinematic | IV static |
| $P_z$ | zf | | $w_n = 0$<br>$w_t = 0$ | $\gamma_n(\mathbf{V}) \leqslant 0$ | |
| | zd | $\gamma_n(\mathbf{V}) = 0$<br>$\gamma_t(\mathbf{V}) = 0$ | | | $w_n \leqslant 0$<br>$\lvert w_t\rvert + \mu w_n \leqslant 0$ |
| | zs+ | $\gamma_n(\mathbf{V}) = 0$ | $w_t - \mu w_n = 0$ | $\gamma_t(\mathbf{V}) \geqslant 0$ | $w_n \leqslant 0$ |
| | zs− | $\gamma_n(\mathbf{V}) = 0$ | $w_t + \mu w_n = 0$ | $\gamma_t(\mathbf{V}) \leqslant 0$ | $w_n \leqslant 0$ |
| $P_s$ | sd | $\gamma_n(\mathbf{V}) = 0$<br>$\gamma_t(\mathbf{V}) = 0$ | | | $\sigma[r_t^0]w_t + \mu w_n \leqslant 0$ |
| | ss | $\gamma_n(\mathbf{V}) = 0$ | $w_t + \sigma[r_t^0]\mu w_n = 0$ | $\sigma[r_t^0]\gamma_t(\mathbf{V}) \leqslant 0$ | |

$$\mathbf{w}_{free} = \begin{bmatrix} \mathbf{w}_{\text{f}} \\ \mathbf{w}_{\text{zf}} \end{bmatrix} = \mathbf{0}, \quad \mathbf{w}_{stick} = \begin{bmatrix} \mathbf{w}_{\text{d}} \\ \mathbf{w}_{\text{zd}} \\ \mathbf{w}_{\text{sd}} \end{bmatrix},$$

$$\mathbf{w}_{\text{t}\ slip} = \begin{bmatrix} \mathbf{w}_{\text{tzs+}} \\ \mathbf{w}_{\text{tzs}-} \\ \mathbf{w}_{\text{tss}} \end{bmatrix} = \mu \mathbf{S}_{slip} \mathbf{w}_{\text{n}\ slip}, \quad \mathbf{w}_{\text{n}\ slip} = \begin{bmatrix} \mathbf{w}_{\text{nzs+}} \\ \mathbf{w}_{\text{nzs}-} \\ \mathbf{w}_{\text{nss}} \end{bmatrix}$$

with

$$\mathbf{S}_{slip} = \text{diag}(\sigma_{\text{p}}), \quad \sigma_{\text{p}} = \begin{cases} +1, & \text{in zs}+, \\ -1, & \text{in zs}-, \\ -\sigma\left[r_{\text{t}}^{0}\right], & \text{in ss.} \end{cases} \tag{4.21}$$

The elimination of the reaction rates $\hat{\mathbf{w}}$ from (4.19) follows the same steps of the procedures used in Section 3 for the elimination of the reactions $\hat{\mathbf{r}}$ from (3.19). An independent set of generalized velocities $\mathbf{V}^*$ is chosen such that $\mathbf{V} = \mathbf{C}^{\text{T}}\mathbf{V}^*$.

For the first procedure, the elimination of the reactions and the projection of the equations on the directions tangential to the constraints yield the reduced eigensystem

$$\left(\lambda^2 \mathbf{M}^* + \mathbf{K}^*\right)\mathbf{V}^* = \mathbf{0}, \tag{4.22}$$

where $\mathbf{M}^*$ is again given by (3.28) and, similarly,

$$\mathbf{K}^* = \mathbf{C}\left[\mathbf{I} - \hat{\mathbf{G}}_{\mu}^{\text{T}} \hat{\mathbf{I}} \mathbf{G}^{\dagger}\right]\mathbf{K}\mathbf{C}^{\text{T}}. \tag{4.23}$$

These effective mass and stiffness matrices are both affected by the friction coefficient and they are in general non-symmetric. These matrices $\mathbf{M}^*$ and $\mathbf{K}^*$ coincide with their homologous in Martins et al. (1998), when the case of a linear elastic system and a flat obstacle is considered, the normal and tangential displacements of the contact candidate particles are used as generalized coordinates (**G** is a Boolean matrix), and *only* near future evolutions zf and ss are considered for *all* the contact particles in $P_{\text{z}}$ and $P_{\text{s}}$, respectively. The first reaction rates are given by

$$\mathbf{w} = \mathbf{G}^{\dagger}\left(\lambda^2 \mathbf{M} + \mathbf{K}\right)\mathbf{C}^{\text{T}}\mathbf{V}^* \tag{4.24}$$

The second procedure yields the eigensystem

$$\left(\lambda^2 \mathbf{M}^{\#} + \mathbf{K}^{\#}\right)\mathbf{V}^* = \mathbf{0}, \tag{4.25}$$

with $\mathbf{M}^{\#}$ given again by (3.34) and

$$\mathbf{K}^{\#} = \mathbf{C}\left[\mathbf{I} - \hat{\mathbf{G}}_{\mu}^{\text{T}}\left(\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_{\mu}^{\text{T}}\right)^{-1}\hat{\mathbf{G}}\mathbf{M}^{-1}\right]\mathbf{K}\mathbf{C}^{\text{T}}. \tag{4.26}$$

The first reaction rates are given by

$$\mathbf{w} = \mathbf{H}_{\mu}\left(\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_{\mu}^{\text{T}}\right)^{-1}\hat{\mathbf{G}}\mathbf{M}^{-1}\mathbf{K}\mathbf{C}^{\text{T}}\mathbf{V}^*. \tag{4.27}$$

As observed earlier, this second procedure can be applied if matrix $\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_{\mu}^{\mathrm{T}}$ is non-singular. The invertibility of matrices $\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_{\mu}^{\mathrm{T}}$ and $\mathbf{M}^{*}$ is related by Lemma 3.8.

In the next proposition we summarize the equivalence between the results of Propositions 4.1 and 4.2 and those that can be obtained from the previous sets of linear eigenproblems of the form (4.22) or (4.25).

**Proposition 4.3.** Let the assumptions of Proposition 4.1 [or of Proposition 4.2] hold. The statements (4.11) and (4.12) [or (4.17)] hold for some *admissible* $\mathbf{V} \neq \mathbf{0}$ and some $\lambda \geqslant 0$ if and only if some of the linear eigenproblems (4.21) [or (4.25)] is solved by the same value of $\lambda$ and some $\mathbf{V}^{*} \neq \mathbf{0}$, with $\mathbf{V} = \mathbf{C}^{\mathrm{T}}\mathbf{V}^{*}$, and the inequalities in columns III and IV of Table 2 are satisfied by that vector $\mathbf{V}$ and the corresponding $\mathbf{w}$ given by (4.24) [or by (4.27)]. □

*4.3. Divergence instability of an equilibrium state of the original nonlinear system*

Under some additional conditions we can conclude that the equilibrium state of the given nonlinear system is also unstable by divergence.

**Proposition 4.4.** Assume that:

(i) the lines of $\mathbf{G}(\mathbf{X})$ are lineraly independent (2.9);
(ii) the equivalent conditions (4.11) and (4.12) in Proposition 4.1 hold for a real strictly positive $\lambda^2$, and, for the corresponding linear eigenproblem (4.22), all other $\lambda^2$ are non-positive real numbers;
(iii) at all the particles in $P_{\mathrm{z}}$ and $P_{\mathrm{s}}$, the inequalities in columns III and IV in Table 2 are satisfied in the *strict* sense by the vectors $\mathbf{V}$ and $\mathbf{w}$ in (4.11) and (4.13);
(iv) the matrices $\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_{\mu}^{\mathrm{T}}$ in (4.26) and $\mathbf{M}^{*}$ in (4.22) are invertible;
(v) $\mathbf{M}(\mathbf{X}), \mathbf{B}(\mathbf{X}, \dot{\mathbf{X}}), \mathbf{G}(\mathbf{X})$ and $\partial\mathbf{G}(\mathbf{X})/\partial\mathbf{X}$ are continuously differentiable in the neighborhood of $(\mathbf{X}, \dot{\mathbf{X}}) = (\mathbf{X}^0, \mathbf{0})$ (4.1).

Then the equilibrium state of the actual nonlinear system corresponding to $\mathbf{X}^0$ and $\mathbf{r}^0$ is dynamically unstable (a divergence instability).

**Sketch of the proof.** A sufficiently small initial perturbation to the equilibrium state of the nonlinear system is given,

$$\begin{aligned} &\mathbf{X}(\tau) \in K_{\mathrm{X}}, \quad \mathbf{X}(\tau) = \mathbf{X}^0 + \alpha(\tau)\mathbf{V} + \mathbf{X}_{\chi} \\ &\dot{\mathbf{X}}(\tau) \in K_{\mathrm{V}}(\mathbf{X}(\tau)), \quad \dot{\mathbf{X}}(\tau) = \alpha(\tau)\lambda\mathbf{V} + \mathbf{V}_{\chi} \\ &\|(\mathbf{X}_{\chi}, \mathbf{V}_{\chi})\| \leqslant C\alpha^2(\tau)\|(\mathbf{V}, \lambda\mathbf{V})\|^2 \end{aligned} \tag{4.28}$$

where $C > 0, \alpha(\tau) > 0$ and, using assumption (ii), $\lambda > 0$ and $\mathbf{V}$ are a solution to (4.11) and (4.12). Note that, due to the possible curvature of the obstacle, the perturbation given to the nonlinear system cannot in general coincide with the perturbation $\alpha(\tau)(\mathbf{V}, \lambda\mathbf{V})$ of the linearized system: this is the reason for the need to introduce the correcting terms $(\mathbf{X}_{\chi}, \mathbf{V}_{\chi})$ in (4.28). The perturbed states of the contact candidate particles correspond to the strictly *free*, *sliding* or *stick* behaviors determined by the directions $\mathbf{V}$ and $\mathbf{w}$ obtained from (4.11) and (4.13) under the assumption (iii). More specifically we have: $\phi^{\mathrm{p}}(\mathbf{X}(\tau)) < 0$, at all the *free* particles, with $\gamma_{\mathrm{n}}^{\mathrm{p}}(\dot{\mathbf{X}}(\tau)) < 0$ at the zf particles; $\phi^{\mathrm{p}}(\mathbf{X}(\tau)) = \gamma_{\mathrm{n}}^{\mathrm{p}}(\dot{\mathbf{X}}(\tau)) = 0$ and $\sigma_{\mathrm{p}}\gamma_{\mathrm{t}}^{\mathrm{p}}(\dot{\mathbf{X}}(\tau)) > 0$, at the *slip* particles (recall (4.21)); and $\mathbf{x}^{\mathrm{p}}(\mathbf{X}(\tau)) - \mathbf{x}^{\mathrm{p}}(\mathbf{X}^0) = \mathbf{G}^{\mathrm{p}}(\mathbf{X}(\tau))\dot{\mathbf{X}}(\tau) = \mathbf{0}$ at the stick particles. Due to the assumed smoothness of the obstacles, the correcting terms $(\mathbf{X}_{\chi}, \mathbf{V}_{\chi})$ can be chosen with an order of magnitude of $\alpha^2(\tau)$.

On the other hand, assumption (iv) guarantees that it is possible to eliminate the reactions and write both the nonlinear and the linearized equations of motion only in terms of the kinematic varibles $\mathbf{X}(t)$ and $\dot{\mathbf{X}}(t)$ (see Appendix B). The system of nonlinear ordinary differential equations obtained after elimination of the reactions has a unique solution and the reactions are continuous functions of time, as a consequence of (iv) and (v) [Lötstedt, 1981; Lötstedt, 1982a; Vidyasagar, 1993).

But assumption (ii) implies that the coefficient matrix of the linearized system has a strictly positive eigenvalue (the value $\lambda > 0$ in (4.28) and in Proposition 4.1). Consequently, it is possible to show (using assumption (v), see Appendix B) that the solution of the nonlinear system with the initial conditions (4.28) grows exponentially. Finally, the conclusion that the equilibrium state is unstable follows by showing (also in Appendix B) that the perturbed solution $(\mathbf{X}(t), \mathbf{r}(t))$ leaves the neighborhood of the equilibrium state before it reaches the boundary of the admissible region of the configuration-reaction space where it is valid. □

### 4.4. Illustrative examples of small dimension

*Example A. The effect of obstacle curvature on (smooth) divergence*. We consider the 2 d.o.f. system represented in Fig. 6, which is a modified version of the one presented by Klarbring (1990a). It consists of a particle of mass $m$ restrained by a system of linear springs. The reference position of the particle corresponds to its equilibrium under no external forces (undeformed springs). The generalized coordinates are the components of the particle displacement $\mathbf{u} = [u_1\ u_2]^{\mathrm{T}}$ measured from that reference state in the fixed orthonormal reference frame whose origin coincides with that reference position. The $2 \times 2$ elastic stiffness matrix $\mathbf{K}^{\mathrm{U}}$ has diagonal elements $K_{11}$ and $K_{22}$ and non-diagonal components $K_{12} = K_{21}$.

The main modification relatively to the system discussed in Klarbring (1990a) is the curvature of the obstacle $\chi = 1/R$ where $R$ is the radius of curvature. The equilibrium state studied hereafter results from the application of an additional constant force $\mathbf{f}^0$ to the particle m such that $r_{\mathrm{t}}^0 = \mu r_{\mathrm{n}}^0$: a situation of impending sliding towards the right. In this case $\mathbf{G}_{\mathrm{t}}(\mathbf{u}^0) = [1\ 0]$ and $\mathbf{G}_{\mathrm{n}}(\mathbf{u}^0) = [0\ 1]$. The sets of admissible

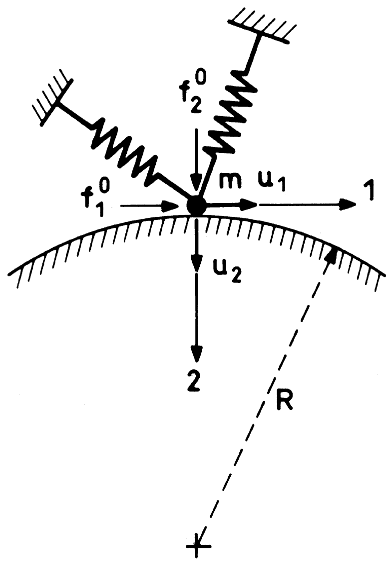


Fig. 6. Modified form of Klarbring's example (Klarbring, 1990a) involving a curved obstacle.

right velocities and right reaction rates are

$$K_V(\mathbf{u}^0, \mathbf{r}^0) = \{(V_1, V_2): V_2 = 0, V_1 \geqslant 0\}$$

$$K_\mathbf{w}(\mathbf{u}^0, \mathbf{r}^0, \mathbf{V}) = \{(w_t, w_n): -w_t + \mu w_n \leqslant 0, (-w_t + \mu w_n)V_1 = 0\}.$$

For this example, eqn (4.22) gives

$$\left[\lambda^2\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} + \begin{bmatrix} K_{11} - \dfrac{f_2^0}{R} & K_{12} \\ K_{12} + \mu\dfrac{f_2^0}{R} & K_{22} \end{bmatrix}\right] \cdot \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} w_t \\ w_n \end{bmatrix}$$

where the tangent stiffness matrix was constructed as indicated in (4.23). Using the fact that $V_2 = 0$ and $w_t = \mu w_n$, the previous system reduces to

$$\left(\lambda^2 m + \left(K_{11} - \mu K_{12} - (1 + \mu^2)\frac{f_2^0}{R}\right)\right)V_1 = 0,$$

the single equation that corresponds to (4.22). When $K_{11} - \mu K_{12} - (1 + \mu^2)f_2^0/R < 0$ we conclude from Proposition 4.4 that the system is unstable by divergence. When the obstacle curvature vanishes the last equation yields the well-known result for the original example of Klarbring (flat obstacle). From the previous equation we conclude that, as expected, the convexity of the obstacle ($\chi = +1/R > 0$) facilitates instability.

*Example B. A non-associative Shanley column.* This example deals with a column having elastic supports that are connected to frictional sliders, as represented in Fig. 7. It is a modified version of another example presented by Klarbring (1998). The model consists of a rigid homogeneous bar AB of length $L$ and mass $M$ rigidly connected at point B to a rigid massless bar orthogonal to AB. Point B has no horizontal displacement. The column is supported by four springs of stiffness $K$ each. The rigid bars are connected with the four vertical springs by frictionless sliders $E$ and $F$. These four springs remain vertical. The particles C and D of mass $M_p$ are attached to the upper extremities of the upper vertical springs and may establish unilateral frictional contact with horizontal obstacles. The coefficient of friction is $\mu$. Particles C and D are also attached to horizontal springs of stiffness $K$. The initial distance of the vertical springs to the axis of symmetry is $L_h$. A vertical downward prescribed displacement $\bar{U}_v$ is applied to both the horizontal obstacles: $\bar{U}_v$ is measured from the equilibrium position of the particles C and D when they are acted only by gravity. The extremities of the horizontal springs opposite to particles C and D have both an initial horizontal prescribed displacement $\bar{U}_h$ which corresponds to an initial compression of those springs.

This mechanical system has six degrees-of-freedom. We choose as generalized coordinates the vertical displacement ($\delta$) of point B, the angle ($\theta$) between bar AB and the vertical and the horizontal ($u_t^C, u_t^D$) and vertical ($u_n^C, u_n^D$) components of the displacements of particles C and D. The displacements are measured from the reference configuration that coincides with the symmetric equilibrium configuration of the system under the simultaneous action of gravity and vertical prescribed displacements ($\bar{U}_v$). $\mathbf{X} = [\delta\ \theta\ u_n^C\ u_n^D\ u_t^C\ u_t^D]^T$ is the vector of non-dimensional generalized coordinates. The generalized coordinates with dimension of length are non-dimensionalized by multiplying them by the factor $1/L$. The non-dimensional prescribed displacements are $\bar{u}_v = \bar{U}_v/L$ and $\bar{u}_h = \bar{U}_h/L$. The non-dimensional external force $P$ and the non-dimensional reaction forces $\mathbf{r} = (r_n^C, r_n^D, r_t^C, r_t^D)$ are obtained from the dimensional ones by multiplication by the factor $1/KL$. Time is non-dimensionalized by multiplication by the factor

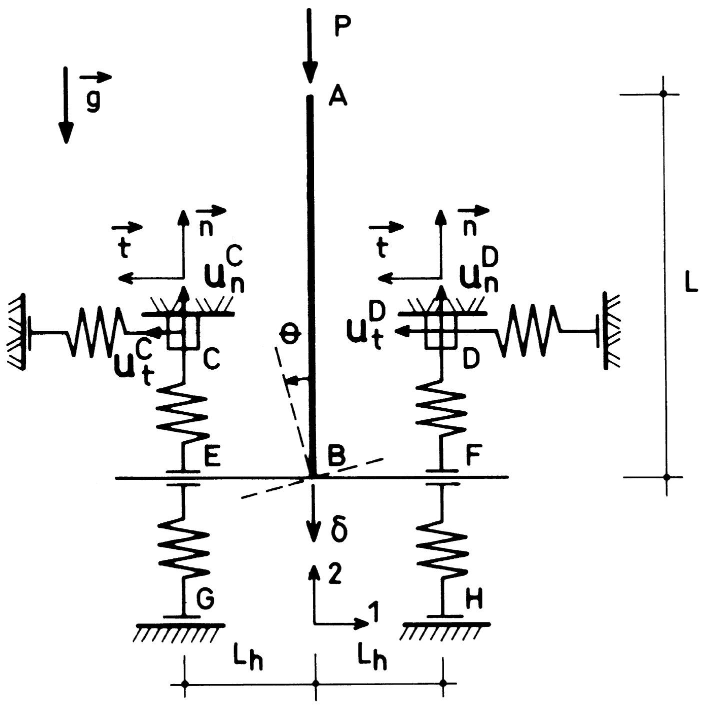


Fig. 7. A column with frictional-contact supports leading to bifurcations from stable configurations of the fundamental trajectory involving sliding of one or both unilateral contacts.

$\sqrt{K/M}$. The horizontal prescribed displacements $\bar{u}_h$ have the upper bound $\mu\bar{u}_v/2$, so as to keep the reference equilibrium reactions at C and D inside the friction cone.

In this particular example the complete (non-dimensional) Jacobian matrix $\mathbf{G}$ and the left inverse $\mathbf{G}^\dagger$ of $\mathbf{G}^T$ are equal:

$$\mathbf{G}^\dagger = \mathbf{G} = \begin{bmatrix} \mathbf{G}_n^C \\ \mathbf{G}_n^D \\ \mathbf{G}_t^C \\ \mathbf{G}_t^D \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The fundamental trajectory of the system is

$$\delta = \frac{P}{4}; \quad \theta = u_n^C = u_n^D = u_t^C = u_t^D = 0, \quad \text{for } P \in [0, P_A[,$$

$$\delta = \frac{P}{4}; \quad \theta = u_n^C = u_n^D = 0, \quad u_t^D = -u_t^C = \bar{u}_h - \frac{\mu}{2}\left(\bar{u}_v - \frac{P}{2}\right), \quad \text{for } P \in [P_A, \quad P_B[$$

$$\delta = \frac{P}{2}; \quad \theta = 0; \quad u_n^C = u_n^D = -\frac{P}{2}; \quad u_t^D = -u_t^C = \bar{u}_h, \quad \text{for } P \geqslant P_B,$$

where $P_A = 2(\bar{u}_v - 2\bar{u}_h/\mu)$ is the (non-dimensional) load at which impending slip of the particles C and D is first attained, and $P_B = 2\bar{u}_v$ is the value of the vertical load for which the reactions at C and D vanish. For a graphical representation of the fundamental trajectory see Fig. 8.

For the set of non-dimensional generalized coordinates defined above the mass and stiffness matrices are

$$\mathbf{M} = \mathrm{diag}\left(1, \tfrac{1}{3}, m, m, m, m\right)$$

and

$$\mathbf{K} = \begin{bmatrix} 4 & 0 & 1 & 1 & 0 & 0 \\ 0 & a & l - u_t^D & -(l - u_t^D) & \dfrac{P}{2} & \dfrac{P}{2} \\ 1 & l - u_t^D & 1 & 0 & 0 & 0 \\ 1 & -(l - u_t^D) & 0 & 1 & 0 & 0 \\ 0 & \dfrac{P}{2} & 0 & 0 & 1 & 0 \\ 0 & \dfrac{P}{2} & 0 & 0 & 0 & 1 \end{bmatrix}$$

for a generic equilibrium state $(\mathbf{X}^0, \mathbf{r}^0)$ on the fundamental trajectory. The non-dimensional parameters that govern the behavior of the system are $m = M_p/M$, $l = L_h/L$, $w = \mathrm{Mg}/(2KL)$ as well as $\bar{u}_v$, $\bar{u}_h$ and $\mu$ defined before. In this study we denote by $a$ and $b$ the following non-dimensional quantities evaluated along the fundamental trajectory: $a = a(u_t^D, P) = 4(l - u_t^D)^2 - w - P$, $b = b(u_t^D, P) = P/2 + \mu(l - u_t^D)$.

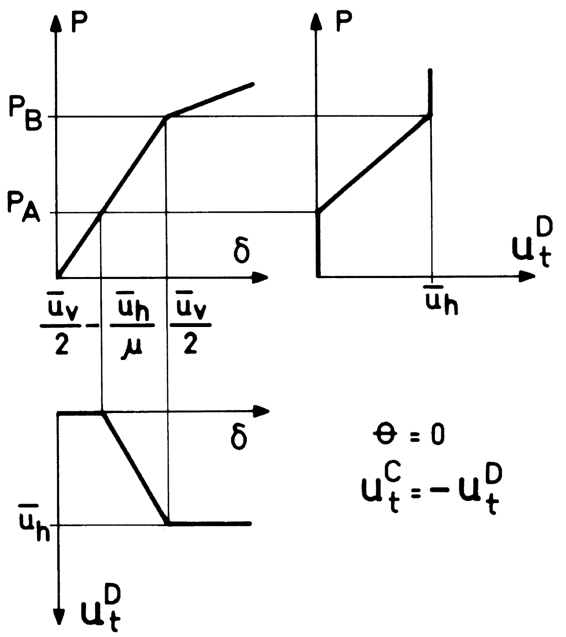


Fig. 8. Orthogonal projections in space $(P, \delta, u_t^D)$ of the fundamental trajectory of the system of Fig. 7. Note that for $P \geqslant P_B$, $u_n^C = u_n^D \equiv -P/2$.

For the given particular type of loading, we wish to determine the quasistatic paths that may bifurcate from the fundamental path and to study the dynamic stability of the fundamental equilibrium states. At some equilibrium state on the fundamental path and for a given load rate, the existence of other quasistatic paths is studied by determining the kinematically admissible displacement rates and the admissible reaction rates (4.8) for which the rate form of the equilibrium eqns (2.18) is satisfied (see Klarbring, 1990a; Nguyen, 1994). The divergence instability of the equilibrium states is studied with the theory presented in Section 4 (Proposition 4.4): the effective mass and stiffness matrices (3.28) and (4.23) are calculated, the linear eigenproblems of the type (4.22) are solved, and the strict inequalities referred in (iii) of Proposition 4.4 are checked.

*Bifurcations from the fundamental path*

At a generic equilibrium state $(\mathbf{X}^0, \mathbf{r}^0)$ along the fundamental trajectory, the rate form of the equilibrium equations is

$$\mathbf{KV} = \mathbf{e}\dot{P} + \mathbf{G}^{\mathrm{T}}\mathbf{w},$$

where

$$\mathbf{V} = \begin{bmatrix} V_\delta & V_\theta & V_{\mathrm{n}}^{\mathrm{C}} & V_{\mathrm{n}}^{\mathrm{D}} & V_{\mathrm{t}}^{\mathrm{C}} & V_{\mathrm{t}}^{\mathrm{D}} \end{bmatrix}^{\mathrm{T}} \in K_{\mathrm{V}}(\mathbf{X}^0, \mathbf{r}^0),$$
$$\mathbf{w} = \begin{bmatrix} w_{\mathrm{n}}^{\mathrm{C}} & w_{\mathrm{n}}^{\mathrm{D}} & w_{\mathrm{t}}^{\mathrm{C}} & w_{\mathrm{t}}^{\mathrm{D}} \end{bmatrix}^{\mathrm{T}} \in K_{\mathbf{w}}(\mathbf{X}^0, \mathbf{r}^0, \mathbf{V})$$
$$\mathbf{e} = -(\partial^2\Omega/\partial\mathbf{X}\partial P)(\mathbf{X}^0) = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$$

denote the vectors of non-dimensional displacement rates, reaction rates and relative magnitudes of the generalized applied force rates, respectively; $\dot{P}$ is the load rate. We consider the following kinds of equilibrium states, for which the admissible sets of right velocities and right reaction rates are presented.

1. reactions at C and D are strictly inside the friction cone ($0 \leqslant P < P_{\mathrm{A}}$)

$$K_{\mathrm{V}}(\mathbf{X}^0, \mathbf{r}^0) = \{\mathbf{V} \in \mathbb{R}^6 \colon V_{\mathrm{n}}^{\mathrm{C}} = V_{\mathrm{n}}^{\mathrm{D}} = V_{\mathrm{t}}^{\mathrm{C}} = V_{\mathrm{t}}^{\mathrm{C}} = 0\}$$
$$K_{\mathbf{w}}(\mathbf{X}^0, \mathbf{r}^0, \mathbf{V}) = \mathbb{R}^4;$$

2. both particles are in a state of impending slip ($P_{\mathrm{A}} \leqslant P < P_{\mathrm{B}}$)

$$K_{\mathrm{V}}(\mathbf{X}^0, \mathbf{r}^0) = \{\mathbf{V} \in \mathbb{R}^6 \colon V_{\mathrm{n}}^{\mathrm{C}} = V_{\mathrm{n}}^{\mathrm{D}} = 0,\ V_{\mathrm{t}}^{\mathrm{C}} \leqslant 0,\ V_{\mathrm{t}}^{\mathrm{D}} \geqslant 0\}$$
$$K_{\mathbf{w}}(\mathbf{X}^0, \mathbf{r}^0, \mathbf{V}) = \{\mathbf{w} \in \mathbb{R}^4 \colon w_{\mathrm{t}}^{\mathrm{C}} + \mu w_{\mathrm{n}}^{\mathrm{C}} \leqslant 0,\ (w_{\mathrm{t}}^{\mathrm{C}} + \mu w_{\mathrm{n}}^{\mathrm{C}})V_{\mathrm{t}}^{\mathrm{C}} = 0$$
$$-w_{\mathrm{t}}^{\mathrm{D}} + \mu w_{\mathrm{n}}^{\mathrm{D}} \leqslant 0, (-w_{\mathrm{t}}^{\mathrm{D}} + \mu w_{\mathrm{n}}^{\mathrm{D}})V_{\mathrm{t}}^{\mathrm{D}} = 0\};$$

3. both particles are in a state of geometric contact without reaction ($P = P_{\mathrm{B}}$)

$$K_{\mathrm{V}}(\mathbf{X}^0, \mathbf{r}^0) = \{\mathbf{V} \in \mathbb{R}^6 \colon V_{\mathrm{n}}^{\mathrm{C}} \leqslant 0,\ V_{\mathrm{t}}^{\mathrm{D}} \leqslant 0\}$$
$$K_{\mathbf{w}}(\mathbf{X}^0, \mathbf{r}^0, \mathbf{V}) = \{\mathbf{w} \in \mathbb{R}^4 \colon w_{\mathrm{n}}^{\mathrm{C}} \leqslant 0,\ w_{\mathrm{n}}^{\mathrm{C}} V_{\mathrm{n}}^{\mathrm{C}} = 0,\ |w_{\mathrm{t}}^{\mathrm{C}}| \leqslant -\mu w_{\mathrm{n}}^{\mathrm{C}}$$
$$w_{\mathrm{n}}^{\mathrm{D}} \leqslant 0,\ w_{\mathrm{t}}^{\mathrm{D}} V_{\mathrm{n}}^{\mathrm{D}} = 0,\ |w_{\mathrm{t}}^{\mathrm{D}}| \leqslant -\mu w_{\mathrm{n}}^{\mathrm{D}}\};$$

4. both particles are out of contact ($P > P_{\mathrm{B}}$)

$$K_{\rm V}(\mathbf{X}^0, \mathbf{r}^0) = \mathbb{R}^6$$

$$K_{\mathbf{w}}(\mathbf{X}^0, \mathbf{r}^0, \mathbf{V}) = \{\mathbf{0}\} \subset \mathbb{R}^4.$$

It is convenient to define the three external loads corresponding to the following conditions involving the quantities $a$ and $b$:

$P = P_1$ when $a = bP$ holds;

$P = P_2$ when $a = bP/2$ holds, and

$P = P_3$ when $a = 0$.

Different kinds of response may occur for different values of the non-dimensional governing parameters. The bifurcation diagram depends qualitatively on the relative position of the two sets of loads $P_{\rm A}$, $P_{\rm B}$ and $P_1, P_2, P_3$. In this paper we only consider the case for which the governing parameters are such that $P_{\rm A} < P_2 < P_3 < P_{\rm B}$. A complete study of this example can be found in Martins and Costa (1998). In the case studied here (and for the particular type of loading considered) the only bifurcations from the fundamental path occur for $P \in [P_{\rm A}, P_{\rm B}[$. Table 3 summarizes the relevant information for the quasistatic evolutions involving rotation of bar AB that branch from the fundamental path. The near future frictional contact states of the two particles C and D are indicated by the use of the words FREE, SLIP or STICK: for instance, a near future evolution involving sliding of C and stiction of D is indicated by SLIP–STICK, whilst STICK–SLIP indicates that C will be stuck and D will be sliding in the near future (the same rule applies for other combinations of states). Table 3 contains, for each case of near future evolution in the range $P \in [P_{\rm A}, P_{\rm B}[$, the active generalized velocities, the effective stiffness matrices, the conditions on the data resulting from the constraints in $K_{\rm V}(\mathbf{X}^0, \mathbf{r}^0)$ and $K_{\mathbf{w}}(\mathbf{X}^0, \mathbf{r}^0, \mathbf{V})$ and the near future (rate) solutions. The conditions on the data are written in terms of the quantities $a$ and $b$ and also in terms of the loads $P_1, P_2, P_3$ (between square brackets).

In Table 3 it can be seen that the fundamental path has bifurcations for a load rate $\dot{P} > 0$ in the following ranges: $[P_1, P_2]$, if $P_1 > P_{\rm A}$; or $[P_{\rm A}, P_2[$, if $P_1 \leqslant P_{\rm A}$. If $P_1 > P_{\rm A}$ the problem in the first rates indicates a bifurcation into SLIP–SLIP solutions at $P_1$: the problem in the first rates has infinitely many SLIP–SLIP solutions which are represented in Fig. 9 by a fan. For $P_1 \geqslant P_{\rm A}$ there exists a continuous range of bifurcation points into a STICK–SLIP or a SLIP–STICK solution with $\dot{P} > 0$. The upper load of this segment is $P_2$. The secondary paths for that load begin with $\dot{P} = 0$ (Fig. 10). For $P \in ]P_2, P_3[$, the bifurcations into STICK–SLIP or SLIP–STICK happen with a load decrease ($\dot{P} < 0$) (Fig. 10). Above $P_3$ and below $P_{\rm B}$ there are no secondary branches from the fundamental path. Table 3 shows only the STICK–SLIP case because the SLIP–STICK solutions are symmetric of the STICK–SLIP solutions. The problem in the first rates gives also a STICK–STICK kind of solution with $\dot{P} < 0$ at $P_3$ (see Table 3). That problem has infinitely many solutions, i.e., at the point of the fundamental trajectory where $a = 0$ there is a fan of first rate solutions leading to STICK–STICK frictional contact states (see Fig. 11).

An observation of utmost importance is that when the coefficient matrix $\mathbf{K}^*$ is singular or when a first rate solution gives ambiguous information on the near future state of some contact particle (at least one inequality in Table 2 does not hold in the strict sense), the first rate solution may not be continued by a solution of the same type of the original nonlinear problem. The (higher-order) post-buckling analysis that is needed in these circumstances falls outside the scope of this paper. The above observation is particularly relevant for the cases in which the first rate problem gives a fan of solutions: in the related problem of the (associated) Shanley column (Shanley, 1947) the problem in the first rates also gives fans of solutions (Petryk, 1993) which however cannot all be continued in the post-buckling range (Hutchinson, 1974; Needleman and Tvergaard, 1982; Bazant and Cedolin, 1991).

Table 3
Active generalized velocities, effective stiffness matrices, conditions on the data and the solutions of the problem in the first rates that involve escape from the fundamental trajectory involving rotation of bar AB for $P \in [P_A, P_B] \cdot a = a(u_t^D, P) = 4(l-u_t^D)^2 - w - P$, $b = b(u_t^D, P) = (P/2) + \mu(l-u_t^D)$, $u_t^D = \bar{u}_h - (\mu/2)[\bar{u}_v - (P/2)]$

| Cases | $\mathbf{V}^*$ | $\mathbf{K}^*$ | Conditions on the data | Solution |
|---|---|---|---|---|
| SLIP–SLIP | $\begin{bmatrix} V_\delta \\ V_\theta \\ V_t^C \\ V_t^D \end{bmatrix}$ | $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & a & \frac{P}{2} & \frac{P}{2} \\ \mu & b & 1 & 0 \\ -\mu & b & 0 & 1 \end{bmatrix}$ <br> $\lvert \mathbf{K}^* \rvert = 4(a-bP)$ | $a = bP[P = P_1]$ and $\dot{P} > 0$ | $V_\delta = \dot{P}/4;\ V_\theta = \left(\frac{\mu\dot{P}}{4} - V_t^D\right)/b;\ V_n^C = V_n^D = 0;$ <br> $V_t^C = V_t^D - \frac{\mu\dot{P}}{2};$ <br> $V_t^D \in \left]0, \frac{\mu\dot{P}}{2}\right[$ arbitrary; <br> $w_n^C = V_\delta + (l-u_t^D)V_\theta;\ w_n^D = V_\delta - (l-u_t^D)V_\theta;$ <br> $w_t^C = -\mu w_n^C;\ w_t^D = \mu w_n^D$ |
| STICK–SLIP | $\begin{bmatrix} V_\delta \\ V_\theta \\ V_t^D \end{bmatrix}$ | $\begin{bmatrix} 4 & 0 & 0 \\ 0 & a & \frac{P}{2} \\ -\mu & b & 1 \end{bmatrix}$ <br> $\lvert \mathbf{K}^* \rvert = 4\left(a - b\frac{P}{2}\right)$ | $0 < a < b\frac{P}{2}[P_2 < P < P_3]$ <br> and $\dot{P} < 0$ <br> or <br> $b\frac{P}{2} < a < bP[P_1 < P < P_2]$ <br> and $\dot{P} > 0$ | $V_\delta = \dot{P}/4;\ V_\theta = \frac{-\mu P\dot{P}}{8\left(a - b\frac{P}{2}\right)};\ V_n^C = V_n^D = V_t^C = 0;$ <br> $V_t^D = \frac{\mu a\dot{P}}{4\left(a - b\frac{P}{2}\right)};$ <br> $w_n^C = V_\delta + (l-u_t^D)V_\theta;\ w_t^C = \frac{P}{2}V_\theta;\ w_n^D = V_\delta - (l-u_t^D)V_\theta;$ <br> $w_t^D = \mu w_n^D$ |
| | | | $a = b\frac{P}{2}[P = P_2]$ and $\dot{P} = 0$ | $V_\delta = 0;\ V_\theta < 0$ arbitrary; $V_n^C = V_n^D = V_t^C = 0;$ <br> $V_t^D = -bV_\theta;$ <br> $w_n^C = (l-u_t^D)V_\theta;\ w_t^C = \frac{P}{2}V_\theta;$ <br> $w_n^D = -(l-u_t^D)V_\theta;\ w_t^D = \mu w_n^D$ |
| STICK–STICK | $\begin{bmatrix} V_\delta \\ V_\theta \end{bmatrix}$ | $\begin{bmatrix} 4 & 0 \\ 0 & a \end{bmatrix}$ <br> $\lvert \mathbf{K}^* \rvert = 4a$ | $a = 0[P = P_3]$ and $\dot{P} < 0$ | $V_\delta = \dot{P}/4;\ V_\theta \in \left[\frac{\mu\dot{P}}{4b}, -\frac{\mu\dot{P}}{4b}\right]$ arbitrary; <br> $V_n^C = V_n^D = V_t^C = V_t^D = 0;$ <br> $w_n^C = V_\delta + (l-u_t^D)V_\theta;$ <br> $w_n^D = V_\delta - (l-u_t^D)V_\theta;$ <br> $w_t^C = \frac{P}{2}V_\theta;\ w_t^D = \frac{P}{2}V_\theta$ |

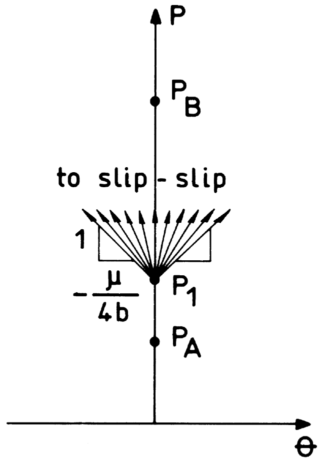


Fig. 9. Fan of rate solutions leading to SLIP–SLIP frictional contact states ($P = P_1$). The qualitative rate solutions are represented in the ($P$, $\theta$) space.

*Divergence instability of the equilibrium states on the fundamental path*

1. $0 \leqslant P < P_A$

In this case the eigenproblem (4.22) has the form

$$\begin{bmatrix} \lambda^2 + 4 & 0 \\ 0 & \dfrac{\lambda^2}{3} + a \end{bmatrix} \begin{bmatrix} V_\delta \\ V_\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $a$ vanishes only for $P = P_3 > P_A$, all the eigenvalues $\lambda$ are pure imaginary: no purely elastic instability occurs in this load range.

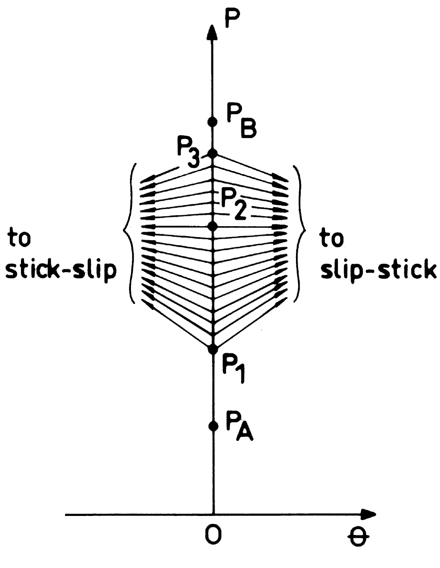


Fig. 10. Rate solutions leading to STICK–SLIP or SLIP–STICK frictional contact states (case $P_A < P_1 < P_2 < P_3 < P_B$).

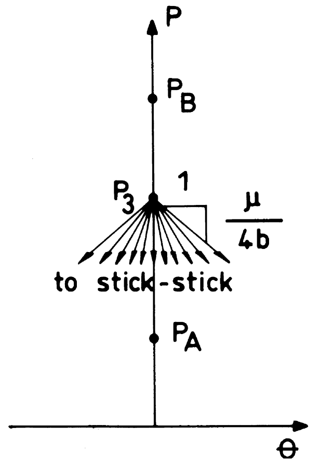


Fig. 11. Fan of rate solutions leading to a STICK–STICK frictional contact state ($P = P_3$).

2. $P_A \leqslant P < P_B$

For the SLIP–SLIP case the eigenproblem (4.22) becomes

$$\begin{bmatrix} \lambda^2+4 & 0 & 0 & 0 \\ 0 & \dfrac{\lambda^2}{3}+a & \dfrac{P}{2} & \dfrac{P}{2} \\ \mu & b & \lambda^2 m+1 & 0 \\ -\mu & b & 0 & \lambda^2 m+1 \end{bmatrix} \begin{bmatrix} V_\delta \\ V_\theta \\ V_t^C \\ V_t^D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the characteristic equation is

$$(\lambda^2+4)(\lambda^2 m+1)\left[\left(\frac{\lambda^2}{3}+a\right)(\lambda^2 m+1) - bP\right] = 0.$$

A sufficient condition for the above equation to have a positive real root is that $a-bP < 0$ which is equivalent to det $(\mathbf{K}^*) < 0$ for the $\mathbf{K}^*$ matrix of the SLIP–SLIP case. However, the eigenvector $\mathbf{V}$ must satisfy $V_t^C = V_t^D$, which is not compatible with the assumed orientations of slip for the particles C and D. Consequently, in this case, the occurrence of divergence instabilities of the type (4.7) is excluded for the linearized system.

For the STICK–SLIP case the eigenproblem to solve is

$$\begin{bmatrix} \lambda^2+4 & 0 & 0 \\ 0 & \dfrac{\lambda^2}{3}+a & \dfrac{P}{2} \\ -\mu & b & \lambda^2 m+1 \end{bmatrix} \begin{bmatrix} V_\delta \\ V_\theta \\ V_t^D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The corresponding characteristic equation is

$$(\lambda^2+4)\left[\left(\frac{\lambda^2}{3}+a\right)(\lambda^2 m+1)-b\frac{P}{2}\right]=0.$$

A necessary and sufficient condition for the existence of one positive $\lambda$ is that $a-b(P/2)<0$ which is equivalent to det $(\mathbf{K}^*)<0$ for the $\mathbf{K}^*$ matrix of this STICK–SLIP case. Since $a-bP/2>0$ for $P<P_2$, then divergence instability of the type (4.7) is excluded for the linearized system in the continuous range of bifurcation points $[P_1, P_2[$. Divergence instability for the case STICK–SLIP may occur for $P \geqslant P_2$: for the actual nonlinear system the equilibrium states of the fundamental trajectory corresponding to $P \in [P_2, P_{\mathrm{B}}[$ are unstable by divergence. These conclusions are also valid for the SLIP–STICK case.

We explore now the possibility of occurrence of a purely elastic instability, i.e. involving stiction of both particles C and D. The eigensystem (4.22) is the same presented for the case $0 \leqslant P < P_{\mathrm{A}}$. The reaction rates are obtained from (4.24),

$$\begin{bmatrix} w_n^{\mathrm{C}} \\ w_{\mathrm{n}}^{\mathrm{D}} \\ w_{\mathrm{t}}^{\mathrm{C}} \\ w_{\mathrm{t}}^{\mathrm{D}} \end{bmatrix} = \begin{bmatrix} (l-u_{\mathrm{t}}^{\mathrm{D}})V_\theta \\ -(l-u_{\mathrm{t}}^{\mathrm{D}})V_\theta \\ \frac{P}{2}V_\theta \\ \frac{P}{2}V_\theta \end{bmatrix},$$

which would lead to reactions at one of the particles outside the friction cone: a purely elastic instability for $P \in [P_{\mathrm{A}}, P_{\mathrm{B}}]$ is not possible.

3. $P = P_{\mathrm{B}}$
A complete analytical study of the dynamic stability of the fundamental equilibrium state corresponding to $P = P_{\mathrm{B}}$ is not done here because it would involve too lengthy computations. A numerical study that includes this case is presented below.
4. $P > P_{\mathrm{B}}$
This case reduces to the study of an elastic stability eigenproblem for a 6 degree-of-freedom system without unilateral contact and friction, with $\mathbf{M}^* = \mathbf{M}$ and $\mathbf{K}^* = \mathbf{K}$. Since $a$ vanishes for some $P \in [P_{\mathrm{A}}, P_{\mathrm{B}}[$, then $a<0$ for $P > P_{\mathrm{B}}$ which implies det $(\mathbf{K})<0$ (see Martins and Costa, 1998). Then the fundamental equilibrium states for $P > P_{\mathrm{B}}$ are unstable.

*A numerical study*

We consider the following non-dimensional parameters: $\mu = 0.2, l = 1.0, \bar{u}_{\mathrm{v}} = 2.0, \bar{u}_{\mathrm{h}} = 0.1, w = 0.2, m = 1$. Particles C and D reach the state of impending slip for $P_{\mathrm{A}} = 2$ and, following the fundamental path, their reactions vanish for $P_{\mathrm{B}} = 4$. For $P \in [P_{\mathrm{A}}, P_2[= [2, 2.23572[$ the fundamental path has a continuous range of bifurcation points to secondary paths with STICK–SLIP and SLIP–STICK frictional contact states (see Table 3) and increasing load $P$. For $P = P_2 = 2.23572$, the bifurcation to the above secondary paths occurs for constant external load. The fundamental equilibrium states between $P_2$ and $P_{\mathrm{B}}$ are unstable by divergence. Since in this example the parameter $a$ vanishes for $P = P_3 = 3.29774 \in ]P_2, P_{\mathrm{B}}[$, then there is a range of external loads ($P \in ]P_3, P_{\mathrm{B}}[$) corresponding to unstable equilibrium states of the fundamental trajectory for which there are no branching to non-symmetric quasistatic secondary paths, for the particular type of loading considered here. The unstable dynamic solutions correspond to STICK–SLIP or SLIP–STICK near future frictional contact states. For the chosen non-dimensional parameters and for the equilibrium state of geometric contact with vanishing reactions ($P = P_B = 4$) there are no secondary quasistatic equilibrium paths emanating from it, for the particular type of loading considered. Moreover, the equilibrium state for $P = P_B = 4$ is unstable by divergence with modes involving FREE–SLIP or SLIP–FREE near future frictional contact states. For $P > 4$ all the fundamental equilibrium states are unstable since det $(\mathbf{K}^*) < 0$ (a pure elastic instability).

## 5. Summary and conclusions

This paper deals with the instability of equilibrium states of mechanical systems in frictional contact with rigid obstacles. We consider systems with a finite number of degrees-of-freedom, elastic nonlinearities and curved obstacles. Two types of instabilities are addressed: a non-smooth type associated with the lack of uniqueness of the accelerations at the equilibrium state, and a smooth type associated with the non-oscillatory growth of arbitrarily small perturbations to the equilibrium state. The same types of instability were considered earlier by Martins et al. (1998) but for linear elastic systems and flat obstacles.

The occurrence of a non-smooth instability (with no initial perturbation) depends on the verification of the following: the necessary conditions for an initial acceleration and reaction discontinuity are satisfied (Propositions 3.1 and 3.3), some regularity assumptions hold, as well as some conditions on the direction of the acceleration and reaction discontinuity (Proposition 3.6).

The study of the smooth divergence instability of a nonlinear system yields an inclusion or variational inequality eigenproblem (Proposition 4.1) (see also Martins and Costa, 1996; Martins et al., 1998). The resolution of one such inclusion eigenproblem is equivalent to the resolution of a set of linear eigenproblems, each of them corresponding to a directional linearization of the original nonlinear system, together with the verification of some inequalities. For the nonlinear system to be unstable it is sufficient that: one of the admissible linearizations of the system corresponds to an unstable equilibrium, some regularity assumptions are valid and some conditions hold on the direction of the displacement and reaction increments that correspond to the linearized unstable solution (Proposition 4.4).

In the instabilities discussed in the present paper, the conditions posed on the escape directions (the initial acceleration and reaction discontinuity in Proposition 3.6 and the displacement and reaction increments in Proposition 4.4) are such that it is possible to guarantee the existence of a smooth near future solution to the original nonlinear dynamic problem: those directions give an unambiguous information on that near future evolution, i.e. they point towards the interior of an admissible region of smooth behavior of the system, and there the effective mass matrix of the system is non-singular. The well-known fact that smooth dynamic solutions to frictional contact problems (continuous velocities, accelerations and reactions) may fail to exist in general circumstances is the main reason for the above restrictions on the sufficient conditions given in Propositions 3.6 and 4.4.

Finally we observe that, as a result of the nature of the mathematical tools used in this work, and also in some relation with the restrictions mentioned above, the main limitation to the instability analyses discussed in this paper is that they do not consider escape solutions that might involve an infinite number of transitions between different frictional contact states in the neighborhood of the equilibrium state.

## Appendix A

**Proof of Lemma 2.5.** First we prove that $\mathbf{A} \in K_A(\mathbf{X}, \mathbf{r}, \mathbf{V})$ implies (2.34). In fact, it is easy to see that the inequality

$$(\mathbf{GA} + \tilde{\mathbf{a}}) \cdot (\mathbf{r}' - \mathbf{r}) \geqslant \sum_{P_{cc}} \big(\gamma_t(\mathbf{A}) + \tilde{a}_t\big)\big(r'_t - r_t\big)$$

holds, because $r'_n = r'_t = r_n = r_t = 0$ in $P_{cf}, \gamma_n(\mathbf{A}) + \tilde{a}_n = 0$ in $P_{cc} \cap (P_d \cup P_s)$, and because $\gamma_n(\mathbf{A}) + \tilde{a}_n \leqslant 0, r_n = 0$ and $r'_n \leqslant 0$ in $P_{cc} \cup P_z$. The inequality (2.34) follows then by observing that

in $P_v$: $\quad r'_t - r_t = \mu\sigma\big[\gamma_t(\mathbf{V})\big]\big(r'_n - r_n\big);$

in $P_0 \cap P_d$: $\quad \gamma_t(\mathbf{A}) + \tilde{a}_t = 0;$

in $P_0 \cap P_z$:
$$\big(\gamma_t(\mathbf{A}) + \tilde{a}_t\big)\big(r'_t - r_t\big) = \big(\gamma_t(\mathbf{A}) + \tilde{a}_t\big)r'_t \geqslant -|\gamma_t(\mathbf{A}) + \tilde{a}_t||r'_t| \geqslant \mu|\gamma_t(\mathbf{A}) + \tilde{a}_t|\big(r'_n - r_n\big);$$

in $P_0 \cap P_s$:
$$\big(\gamma_t(\mathbf{A}) + \tilde{a}_t\big)\big(r'_t - r_t\big) = -|\gamma_t(\mathbf{A}) + \tilde{a}_t|\sigma[r_t]\big(r'_t - r_t\big) \geqslant |\gamma_t(\mathbf{A}) + \tilde{a}_t|\big(-|r'_t| + |r_t|\big) \geqslant \mu|\gamma_t(\mathbf{A}) + \tilde{a}_t|\big(r'_n - r_n\big).$$

We prove now that (2.34) implies $\mathbf{A} \in K_A(\mathbf{X}, \mathbf{r}, \mathbf{V})$. The proof of the properties of $\gamma_t(\mathbf{A}) + \tilde{a}_t$ in $P_0 \cap P_d$ is trivial. In order to prove that $\gamma_n(\mathbf{A}) + \tilde{a}_n \leqslant 0$ and $(\gamma_n(\mathbf{A}) + \tilde{a}_n)r_n = 0$ in $P_v$, we let $r'_n - r_n = r'_t - r_t = 0$ for all particles different from some particle p in that set. For that particle we get from (2.24) and (2.34)

$$r_n \leqslant 0, \quad r_t = \mu r_n \sigma\big[\gamma_t(\mathbf{V})\big],$$

and

$$\big(\gamma_t(\mathbf{A}) + \tilde{a}_t\big)\big(r'_t - r_t\big) + \big(\gamma_n(\mathbf{A}) + \tilde{a}_n\big)\big(r'_n - r_n\big) \geqslant \mu\sigma\big[\gamma_t(\mathbf{V})\big]\big(\gamma_t(\mathbf{A}) + \tilde{a}_t\big)\big(r'_n - r_n\big),$$

for all $(r'_n, r'_t)$ such that $r'_n \leqslant 0$ and $r'_t = \mu r'_n \sigma[\gamma_t(\mathbf{V})]$. Consequently,

$$\big(\gamma_n(\mathbf{A}) + \tilde{a}_n\big)\big(r'_n - r_n\big) \geqslant 0, \text{ for all } r'_n \leqslant 0,$$

which implies the desired result. In order to prove the same result $[\gamma_n(\mathbf{A}) + \tilde{a}_n \leqslant 0$ and $(\gamma_n(\mathbf{A}) + \tilde{a}_n)r_n = 0]$ in $P_0$, we let $r'_n - r_n = r'_t - r_t = 0$ for all particles different from some particle p in that set. For that particle we get from (2.24) and (2.34)

$$r_n \leqslant 0, \quad |r_t| \leqslant -\mu r_n,$$

and

$$\big(\gamma_t(\mathbf{A}) + \tilde{a}_t\big)\big(r'_t - r_t\big) + \big(\gamma_n(\mathbf{A}) + \tilde{a}_n\big)\big(r'_n - r_n\big) \geqslant \mu|\gamma_t(\mathbf{A}) + \tilde{a}_t|\big(r'_n - r_n\big),$$

for all $(r'_n, r'_t)$ such that $r'_n \leqslant 0$ and $|r'_t| \leqslant -\mu r'_n$. Consequently, we have again

$$\begin{aligned}\big(\gamma_n(\mathbf{A}) + \tilde{a}_n\big)\big(r'_n - r_n\big) &\geqslant \mu|\gamma_t(\mathbf{A}) + \tilde{a}_t|\big(r'_n - r_n\big) - \big(\gamma_t(\mathbf{A}) + \tilde{a}_t\big)\big(r'_t - r_t\big) \\ &\geqslant \mu|\gamma_t(\mathbf{A}) + \tilde{a}_t|\big(r'_n - r_n\big) - \big(\gamma_t(\mathbf{A}) + \tilde{a}_t\big)r'_t + \mu|\gamma_t(\mathbf{A}) + \tilde{a}_t|r_n \\ &= \mu|\gamma_t(\mathbf{A}) + \tilde{a}_t|r'_n - \big(\gamma_t(\mathbf{A}) + \tilde{a}_t\big)r'_t \\ &= 0, \text{ for all } r'_n \leqslant 0,\end{aligned}$$

because either $\gamma_t(\mathbf{A}) + \tilde{a}_t = 0$ or $\gamma_t(\mathbf{A}) + \tilde{a}_t \neq 0$ and, for each $r'_n \leqslant 0, r'_t$ can be chosen with the value $r'_t = \mu\sigma[\gamma_t(\mathbf{A}) + \tilde{a}_t]r'_n$, for which $|\mu\sigma[\gamma_t(\mathbf{A}) + \tilde{a}_t]r'_n| = |r'_t| \leqslant -\mu r'_n$. Finally, the proof that $\gamma_t(\mathbf{A}) + \tilde{a}_t = -|\gamma_t(\mathbf{A}) + \tilde{a}_t|\sigma[r_t]$ in $P_0 \cap P_s$ is done by letting $r'_n - r_n = r'_t - r_t = 0$ in (2.34) for all particles, except for $r'_t - r_t$ of some particle p in $P_0 \cap P_s$. For that particle we get from (2.34)

$$\big(\gamma_t(\mathbf{A}) + \tilde{a}_t\big)\big(r_t' - r_t\big) = \big(\gamma_t(\mathbf{A}) + \tilde{a}_t\big)\sigma[r_t]\big(\sigma[r_t]r_t' - |r_t|\big) \geqslant 0,$$

for all $r_t'$ such that [cf. (2.24)] $|r_t'| \leqslant -\mu r_n' = -\mu r_n = |r_t|$. The desired result follows from the fact that $\sigma[r_t]r_t' - |r_t'| \leqslant |r_t| - |r_t| \leqslant 0$. □

## Appendix B

**Mathematical details of the Proof of Proposition 4.4.** The only combinations of near future states that are relevant for the study of the instability of the nonlinear system are those corresponding to the instability of the linearized system, i.e. those that satisfy the assumptions (ii) and (iii) of Proposition 4.4. In the sequel, we use for the reaction vector **r** and for the kinematic matrix **G** the same partitions that are indicated in (3.18). In other words, a solution of the nonlinear system (3.19) and (3.20) is sought such that the conditions in Table 1, with $\gamma_t(\mathbf{A})$ replaced by $\gamma_t(\mathbf{V})$, hold (compare also with Table 2 and recall the related nomenclature).

Since, by assumption (iv), the matrix $\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_\mu^{\mathrm{T}}$ is regular in the neighborhood of the equilibrium position, then, in the admissible region of smooth behavior determined by the vectors **V** and **w** in (4.11) and (4.13), the nonlinear equations of motion (2.13) can be written as a system of $2N$ first-order ordinary differential equations on $\mathbf{y}(t) = [\mathbf{y}_1^{\mathrm{T}}(t) \quad \mathbf{y}_2^{\mathrm{T}}(t)]^{\mathrm{T}} = [(\mathbf{X}(t) - \mathbf{X}^0)^{\mathrm{T}} \quad \dot{\mathbf{X}}^{\mathrm{T}}(t)]^{\mathrm{T}}$

$$\begin{bmatrix} \dot{\mathbf{y}}_1(t) \\ \dot{\mathbf{y}}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{y}_2(t) \\ -\mathbf{M}^{-1}\bar{\mathbf{B}}\big(\mathbf{y}_1(t), \mathbf{y}_2(t)\big) \end{bmatrix}, \tag{B.1}$$

while the inequalities in columns III and IV of Table 1 hold in the *strict* sense (with $\gamma_t(\mathbf{A})$ replaced by $\gamma_t(\mathbf{V})$). $\bar{\mathbf{B}}$ is given by (3.32) and the reactions are obtained by

$$\mathbf{r}(t) = \mathbf{H}_\mu\hat{\mathbf{r}}(t) = \mathbf{H}_\mu\Big(\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_\mu^{\mathrm{T}}\Big)^{-1}\Big(\hat{\mathbf{G}}\mathbf{M}^{-1}\mathbf{B} - \hat{\dot{\mathbf{G}}}\mathbf{y}_2\Big). \tag{B.2}$$

On the other hand, the linearization (4.3) of the equations of motion in the same admissible region, i.e. the linearized form of (B.1), can be written as an homogeneous system of $2N$ first-order ordinary differential equations

$$\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t), \tag{B.3}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{M}^{-1}\Big[\hat{\mathbf{G}}_\mu^{\mathrm{T}}\Big(\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_\mu^{\mathrm{T}}\Big)^{-1}\hat{\mathbf{G}}\mathbf{M}^{-1} - \mathrm{I}\Big]\mathbf{K} & \mathbf{0} \end{bmatrix}_{(\mathbf{X}^0)} \tag{B.4}$$

and the corresponding reaction increments $\delta\mathbf{r}(t)$ are given by

$$\delta\mathbf{r}(t) = \mathbf{H}_\mu\Big(\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_\mu^{\mathrm{T}}\Big)^{-1}\hat{\mathbf{G}}\mathbf{M}^{-1}\mathbf{K}\mathbf{y}_1(t). \tag{B.5}$$

The nonlinear system (B.1) may be written in the equivalent forms

$$\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{Y}\big(\mathbf{y}(t)\big) \in \mathbb{R}^{2N}, \quad t \geqslant \tau, \tag{B.6}$$

or

$$\dot{\mathbf{z}}(t) = \mathbf{J}\mathbf{z}(t) + \mathbf{Z}(\mathbf{z}(t)) \in \mathbb{R}^{2N}, \quad t \geqslant \tau, \tag{B.7}$$

where $\mathbf{A}$ is the constant $2N \times 2N$ real matrix in (4.32), $\mathbf{J}$ is the real Jordan decomposition of matrix $\mathbf{A}$, $\mathbf{Y}(\mathbf{0}) = \mathbf{Z}(\mathbf{0}) = \mathbf{0}$, $\mathbf{Z}(\mathbf{z}) = \mathbf{S}^{-1}\mathbf{Y}(\mathbf{y})$ and $\mathbf{y} = \mathbf{S}\mathbf{z}$, with $\mathbf{S}$ a regular transformation matrix with real entries. Moreover, $\mathbf{Y}(\mathbf{y})$ and $\mathbf{Z}(\mathbf{z})$ are bounded and at least locally Lipschitz continuous in the neighborhood of $\mathbf{y} = \mathbf{0}$ or $\mathbf{z} = \mathbf{0}$, due to the general assumption (4.1). In addition,

$$\lim_{\|\mathbf{y}\| \to 0} \frac{\|\mathbf{Y}(\mathbf{y})\|}{\|\mathbf{y}\|} = \lim_{\|\mathbf{z}\| \to 0} \frac{\|\mathbf{Z}(\mathbf{z})\|}{\|\mathbf{z}\|} = 0. \tag{B.8}$$

In the present case the real Jordan decomposition of $\mathbf{A}$ has the form

$$\mathbf{J} = \begin{bmatrix} \lambda & 0 & 0 & \dots & 0 \\ 0 & -\lambda & 0 & \dots & 0 \\ 0 & 0 & \mathbf{C}(\lambda_3) & & \\ 0 & 0 & & & \mathbf{C}(\lambda_{\mathrm{I}}) \end{bmatrix},$$

where I denotes the total number of independent eigenvectors of $\mathbf{A}$ and, for $i \geqslant 3$,

$$\mathbf{C}(\lambda_i) = \begin{bmatrix} 0 & \mathrm{Im}\,(\lambda_i) & & & & & \dots \\ -\mathrm{Im}\,(\lambda_i) & 0 & & & & & \dots \\ \gamma & 0 & 0 & \mathrm{Im}\,(\lambda_i) & & & \dots \\ 0 & \gamma & -\mathrm{Im}\,(\lambda_i) & 0 & & & \\ & & \gamma & 0 & 0 & \mathrm{Im}(\lambda_i) & \dots \\ & & 0 & \gamma & -\mathrm{Im}(\gamma_i) & 0 & \dots \\ & & & \dots & & & \dots \end{bmatrix}.$$

The elements $\gamma$ may be taken as strictly positive and their magnitude can be controlled by an adequate choice of the transformation matrix $\mathbf{S}$ (Horn and Johnson, 1985; Coddington and Levinson, 1955; Cronin, 1980; Lütkepohl, 1996). The system (B.7) is of the type

$$\dot{z}_1(t) = \lambda z_1(t) + Z_1(\mathbf{z}(t))$$

$$\dot{z}_2(t) = -\lambda z_2(t) + Z_2(\mathbf{z}(t))$$

$$\dot{z}_{2k+1}(t) = \mathrm{Im}\,(\lambda_{k+2}) z_{2k+1}(t) + \gamma_{2k+1} z_{2k-1}(t) + Z_{2k+1}(\mathbf{z}(t))$$

$$\dot{z}_{2k+2}(t) = -\mathrm{Im}\,(\lambda_{k+2}) z_{2k+2}(t) + \gamma_{2k+2} z_{2k}(t) + Z_{2k+2}(\mathbf{z}(t))$$

where $\gamma_{2k+1}$ and $\gamma_{2k+2}$ are 0 or $\gamma > 0$, and the index $k = 1, \dots, N-1$ is not summed. The essence of the proof lies in the time differentiation of a weighted sum of the quantities

$$R^2(t) = |z_1(t)|^2, \quad r^2(t) = \sum_{i=2}^{2N} |z_i(t)|^2$$

in the neighborhood of the trivial solution (Amann, 1990; Coddington and Levinson, 1955; Cronin, 1980; Verhulst, 1980). From (B.8) we have that

$$\forall \varepsilon > 0 \quad \exists \delta > 0: \quad \forall i = 1, \ldots, 2N: \|\mathbf{z}\| < \delta \Longrightarrow |Z_i(\mathbf{z})| \leqslant \varepsilon |z_i|. \tag{B.9}$$

For $b^2 > 0$, it follows from (B.9) that the following inequality holds:

$$\forall \varepsilon > 0 \quad \exists \delta > 0: \quad \|\mathbf{z}(t)\| < \delta \Longrightarrow \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(R^2 - b^2 r^2)(t) \geqslant (\lambda - \gamma - \varepsilon)|z_1(t)|^2 - b^2(\gamma + \varepsilon)\sum_{i=2}^{2N} |z_i(t)|^2.$$

Choosing $\gamma$ and $\varepsilon$ such that $\gamma + \varepsilon \in [0, \lambda/2]$, then

$$\exists \delta > 0: \quad \|\mathbf{z}(t)\| < \delta \Longrightarrow \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(R^2 - b^2 r^2)(t) \geqslant (\gamma + \varepsilon)(R^2 - b^2 r^2)(t).$$

For $b > 0$ and for $\alpha(\tau)$ sufficiently small, there is always an $a^2 > 0$ such that, for the initial conditions (4.28), $(R^2 - b^2 r^2)(\tau) = a^2$. Let $\mathbf{y}_\mathrm{L}(\tau) = \alpha(\tau)(\mathbf{V}, \lambda\mathbf{V})$ be the initial perturbation to the linearized system along the direction $(\mathbf{V}, \lambda\mathbf{V})$ in the phase space associated with the positive eigenvalue $\lambda > 0$, and let $\mathbf{y}_\chi(\tau) = (\mathbf{X}_\chi, \mathbf{V}_\chi)$ be the correction to $\mathbf{y}_\mathrm{L}(\tau)$ due to the curvature of the obstacle. From (4.28) $\|\mathbf{y}_\chi(\tau)\| \leqslant C\alpha(\tau)^2\|\mathbf{V}, \lambda\mathbf{V}\|^2$ and, since $\mathbf{z} = \mathbf{S}^{-1}\mathbf{y}$, then $|z_{\chi i}(\tau)| \leqslant C'\alpha(\tau)^2$ for $i = 1, \ldots, 2N$ and some $C' > 0$. Note that the only non-vanishing component of $\mathbf{z}_\mathrm{L}(\tau)$ is the one related with $\lambda$, i.e. $z_{\mathrm{L}1}(\tau)$. Using the above estimates and letting $\mathbf{z}(\tau) = \mathbf{z}_\mathrm{L}(\tau) + \mathbf{z}_\chi(\tau)$, the condition $(R^2 - b^2 r^2)(\tau) > 0$ is satisfied if

$$\alpha^2(\tau)\big(1 - C_1\alpha(\tau) - b^2 C_2 \alpha^2(\tau)\big) > 0,$$

where $C_1$ and $C_2$ are positive constants that depend on the matrix $\mathbf{S}$. The above inequality is satisfied for $\alpha(\tau) < (-C_1 + (C_1^2 + 4b^2 C_2)^{1/2})/(2b^2 C_2)$.

Taking initial conditions $\mathbf{z}(\tau) = \mathbf{S}^{-1}\mathbf{y}(\tau)$ such that $\|\mathbf{z}(\tau)\| < \delta$ and $(R^2 - b^2 r^2)(\tau) = a^2 > 0$, we get, by Gronwall's inequality (Amann, 1990),

$$\|\mathbf{z}(t)\| < \delta \Longrightarrow (R^2 - b^2 r^2)(t) \geqslant (R^2 - b^2 r^2)(\tau) \exp\left[2(\gamma + \varepsilon)(t - \tau)\right] \geqslant (R^2 - b^2 r^2)(\tau). \tag{B.10}$$

From (B.10) and from the fact that

$$\|\mathbf{z}(t)\|^2 = \big(R^2 + r^2\big)(t) \geqslant (R^2 - b^2 r^2)(t), \tag{B.11}$$

we conclude that the solution $\mathbf{z}(t)$ leaves the neighborhood $\delta$ of the trivial solution, no matter how small the perturbation $\mathbf{z}(\tau)$ is.

Besides, from (B.10) and (B.11) we may extract additional information on the evolution of the perturbed solutions. Notice that

$$(R^2 - b^2 r^2)(\tau) > 0 \tag{B.12}$$

is equivalent to

$$\mathbf{z} \cdot \mathbf{z} < \left(1 + \frac{1}{b^2}\right)(\mathbf{z} \cdot \mathbf{e}_\mathbf{z})^2, \tag{B.13}$$

where $\mathbf{e}_\mathrm{z} = (1, 0, \ldots, 0) = \mathbf{S}^{-1}\mathbf{y}_\mathrm{L}(\tau)/\|\mathbf{S}^{-1}\mathbf{y}_\mathrm{L}(\tau)\|$ defines the direction in the $\mathbf{z}$ phase space that corresponds to the eigenvector of the coefficient matrix of the linearized system associated with the positive eigenvalue $\lambda > 0$. By means of the transformation $\mathbf{z} = \mathbf{S}^{-1}\mathbf{y}$, the inequality (B.13) also corresponds to the interior of a convex cone that encloses $\mathbf{y}_L(\tau) = \alpha(\tau)(\mathbf{V}, \lambda\mathbf{V})$ in the $\mathbf{y}$ phase space:

$$\mathbf{S}^{-T}\mathbf{S}^{-1}\mathbf{y}\cdot\mathbf{y} < \left(1+\frac{1}{b^2}\right)\left(\mathbf{S}^{-T}\mathbf{S}^{-1}\mathbf{y}\cdot\mathbf{e}_y\right)^2 \tag{B.14}$$

where $\mathbf{e}_y = \mathbf{S}\mathbf{e}_z$ has length one in the norm induced by $\mathbf{S}^{-T}\mathbf{S}^{-1}$. The nature of the spectrum of $\mathbf{A}$ (recall assumption (ii) in Proposition 4.4) guarantees that this cone does not contain any eigenvector associated with any other eigenvalue with positive real part. The aperture of the cones defined by (B.13) and (B.14) decreases when larger values of $b^2$ are considered. Moreover,

$$(R^2 - b^2r^2)(t) \geqslant (R^2 - b^2r^2)(\tau) = a^2 \tag{B.15}$$

defines escape regions in the interior of those cones. The regions mentioned above have a simple representation in the $(R, r)$ space (see Fig. 12). Region E is bounded by the hyperbola $(R^2 - b^2r^2)(t) = a^2$ which is interior to the conic region defined by $(R^2 - b^2r^2)(t) \geqslant 0$. According to (B.10) and (B.11) it is clear that every solution to the system (B.1) with an initial condition simultaneously inside the $\delta$-neighborhood of the trivial solution $\mathbf{z}(t) \equiv \mathbf{0}$ and inside the region E will leave the $\delta$-neighborhood before leaving the region E. Since matrix $\mathbf{S}$ represents a non-singular linear and real transformation between $\mathbf{y}$ and $\mathbf{z}$, what was said about the qualitative evolution of the solutions in terms of the $\mathbf{z}$ coordinates can be easily adapted with minor changes to the $\mathbf{y}$ coordinates (the actual phase-space coordinates). The $\delta$-neighborhood of $\mathbf{z}(t) \equiv \mathbf{0}$ corresponds also to a (convex) neighborhood $U$ of $\mathbf{y}(t) \equiv \mathbf{0}$ (Hadley, 1980). By choosing $\delta$ and $\alpha(\tau)$ sufficiently small and a sufficiently large $b^2$, the apertures of (B.13) and (B.14) are small enough that $(\mathbf{X}(t), \dot{\mathbf{X}}(t))$ remains in the interior of the admissible region of smooth behavior of the system before crossing the border of $U$.

To conclude the proof of instability of the system we need to show that the reactions also remain in the interior of the corresponding admissible region of smooth behavior while the phase space trajectory is inside the neighborhood $U$. If we denote $\boldsymbol{\rho} = (\mathbf{r} - \mathbf{r}^0, \dot{\mathbf{r}})$ where $\mathbf{r}$ is given by (B.2) and $\dot{\mathbf{r}}$ can be

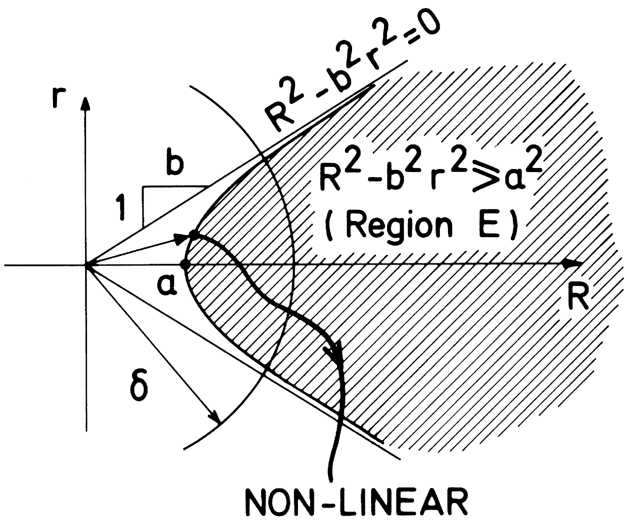


Fig. 12. Qualitative evolution in the space $(R, r)$ of the representative point of a solution of (B.7) (or (B.6)) for a perturbation from equilibrium state given by (4.28).

computed by differentiating (B.2) and using (B.1), we may write

$$\boldsymbol{\rho} = \mathbf{LSz} + \mathbf{F}(\mathbf{z}) \tag{B.16}$$

where

$$\mathbf{L} = \begin{bmatrix} \mathbf{H}_\mu \left(\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_\mu^{\mathrm{T}}\right)^{-1}\hat{\mathbf{G}}\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_\mu \left(\hat{\mathbf{G}}\mathbf{M}^{-1}\hat{\mathbf{G}}_\mu^{\mathrm{T}}\right)^{-1}\hat{\mathbf{G}}\mathbf{M}^{-1}\mathbf{K} \end{bmatrix}.$$

The mapping $\mathbf{LS}$ of the linear part of (B.16) leads to a cone in the $\boldsymbol{\rho}$ space that encloses $(\mathbf{w}, \lambda\mathbf{w})$, where $\mathbf{w}$ is the reaction rate vector of the linearized system corresponding to the positive real eigenvalue $\lambda$. Considering the linear part of (B.16) we have $\mathbf{z} = (\mathbf{LS})^{\mathrm{I}}\boldsymbol{\rho}$, where $(\mathbf{LS})^{\mathrm{I}}$: $Rg(\mathbf{LS}) \rightarrow [Ker(\mathbf{LS})]^{\perp} = Rg((\mathbf{LS})^{\mathrm{T}})$ represents a right inverse of $\mathbf{LS}$ (Lancaster and Tismenetsky, 1985). Then we get from (B.13) that

$$\left((\mathbf{LS})^{\mathrm{I}}\right)^{\mathrm{T}}(\mathbf{LS})^{\mathrm{I}}\boldsymbol{\rho}\cdot\boldsymbol{\rho} \; < \; \left(1 + \frac{1}{b^2}\right)\left(\left((\mathbf{LS})^{\mathrm{I}}\right)^{\mathrm{T}}(\mathbf{LS})^{\mathrm{I}}\boldsymbol{\rho}\cdot\mathbf{e}_\rho\right)^2, \tag{B.17}$$

where $\mathbf{e}_{\boldsymbol{\rho}} = \mathbf{LSe}_{\mathbf{z}}$ is the unit vector along the direction of $(\mathbf{w}, \lambda\mathbf{w})$ (for the norm induced in $Rg(\mathbf{LS})$) by $((\mathbf{LS})^{\mathrm{I}})^{\mathrm{T}}(\mathbf{LS})^{\mathrm{I}})$. The previous inequality defines a cone enclosing $\mathbf{e}_\rho$. The inner product induced by $((\mathbf{LS})^{\mathrm{I}})^{\mathrm{T}}(\mathbf{LS})^{\mathrm{I}}$ in $Rg(\mathbf{LS})$ is used in (B.17). Notice that $((\mathbf{LS})^{\mathrm{I}})^{\mathrm{T}}(\mathbf{LS})^{\mathrm{I}}\boldsymbol{\rho}\cdot\boldsymbol{\rho} = \mathbf{z}\cdot\mathbf{z} \geqslant 0$ for every $\rho \in Rg(\mathbf{LS})$ and that $\mathbf{0} = ((\mathbf{LS})^{\mathrm{I}})^{\mathrm{T}}(\mathbf{LS})^{\mathrm{I}}\boldsymbol{\rho}\cdot\boldsymbol{\rho} = \mathbf{z}\cdot\mathbf{z}$ implies $\mathbf{0} = \mathbf{z} \in [Ker(\mathbf{LS})]^{\perp}$ and $\boldsymbol{\rho} = \mathbf{LSz} = \mathbf{0}$. Due to the presence of higher-order terms $\mathbf{F}(\mathbf{z})$ in (B.16), the image in the $\boldsymbol{\rho}$ space of the phase space cones (B.13) and (B.14) is not the cone (B.17) but it differs from (B.17) by second-order terms. By choosing $\delta$ and $\alpha(\tau)$ sufficiently small and $b^2$ sufficiently large, then for $\mathbf{z}$ in the $\delta$-neighborhood, the region in the $\rho$ space approximated by (B.17) will be strictly inside the admissible region of smooth behavior of the system. In other words, while the variables of the $\mathbf{y}$ phase space are in the neighborhood $U$, the strict inequalities in columns III and IV of Table 1 (with $\gamma_{\mathrm{t}}(\mathbf{A})$ replaced by $\gamma_{\mathrm{t}}(\mathbf{V})$) hold. This shows the validity of the equations of motion of the system at least until it leaves the neighborhood $U$ of equilibrium.

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